Probability

**Definition:** A *probability experiment* is any observable process with well-defined results called *outcomes*. The *sample space* is the set of all possible outcomes. An *event* is a particular outcome or set of outcomes.

**Example 1:** Suppose we are flipping a coin twice. If we denote a “head” by $H$ and a “tail” by $T$, then the sample space is $\{HH, HT, TH, TT\}$. These are all possible outcomes.

**Classical Probability**

In classical probability, every possible outcome is equally likely. Coin-flipping, rolling a single die, and choosing a card from a deck are examples.

**Definition:** Let $S$ be the sample space. The *probability* of event $E$ occurring is given by:

$$P(E) = \frac{\text{Number of outcomes in } E}{\text{Total number of outcomes}} = \frac{n(E)}{n(S)}.$$

**Example 2:** Suppose you choose a card from a standard deck of cards. Find the probability that you’ll get (a) a club, (b) a king, (c) a card numbered less than 5.

There are 52 cards in a standard deck of cards, so $n(S) = 52$. For (a), $n(\text{club}) = 13$, so the probability of getting a club is $P(\text{club}) = \frac{13}{52} = \frac{1}{4}$. For (b), $n(\text{king}) = 4$, so $P(\text{king}) = \frac{4}{52} = \frac{1}{13}$. For (c) $n(<5) = 12$, so $P(<5) = \frac{12}{52} = \frac{3}{13}$.

**Example 3:** What is the probability of each event on a single roll of a die?

(a) You get a 5.  
(b) You get an odd number.  
(c) You get a number less than 7.  
(d) You get a number less than or equal to 4.

(a) $P(5) = \frac{5}{6}$  
(b) $P(\text{odd}) = \frac{3}{6} = \frac{1}{2}$  
(c) $P(<7) = 1$  
(d) $P(\leq 4) = \frac{4}{6} = \frac{2}{3}$
Note: We will usually express probabilities in reduced fraction form. Decimals are also allowed. Sometimes probabilities are also expressed as percentages, although this is discouraged in general.

**Definition**: Let $E$ be an event (i.e. a set of outcomes). The *complement* of $E$, denoted by $\overline{E}$, is the set of all outcomes not included in $E$.

**Example 4**: Find the complement of each event:
(a) Choosing a fruit from a bowl of apples, oranges and bananas and getting a banana.
(b) Rolling a die and getting an odd number.
(c) Drawing a card and getting a face card.

(a) Getting an apple or an orange. (b) Getting an even number. (c) Getting an ace or a numbered card.

**Facts about Probabilities**:
(1) For any event $E$, $0 \leq P(E) \leq 1$.
(2) If an event is impossible (i.e. there are no outcomes in $E$), then $P(E) = 0$.
(3) If an event is certain (i.e. every outcome is in $E$), then $P(E) = 1$.
(4) The sum of the probabilities for all outcomes in the sample space is 1.
(5) $P(\overline{E}) = 1 - P(E)$.

**Example 5**: If the probability that a person owns a mp3 player is $\frac{3}{10}$, find the probability that a person does not own a mp3 player.

Since owning an mp3 player and not owning an mp3 player are complementary events,

$$P(\text{not owning an mp3 player}) = 1 - P(\text{owning an mp3 player})$$

$$= 1 - \frac{3}{10}$$

$$= \frac{7}{10}$$

**Empirical Probability**

In empirical probability, the different outcomes are not necessarily equally likely. Suppose we perform an experiment (hence “empirical”) and flip a coin
1000 times. Classical probability says that we should get 500 heads and 500 tails, but that is very unlikely to work out exactly that way. Our actual results could be summarized by a frequency distribution.

<table>
<thead>
<tr>
<th>Class</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>Head</td>
<td>523</td>
</tr>
<tr>
<td>Tail</td>
<td>477</td>
</tr>
</tbody>
</table>

1000

So what is the probability that a randomly selected flip will be a head? It’s \( \frac{523}{1000} \).

So empirical probability is based on observations.

**Example 6:** For a recent year, 51% of families in the United States had no children under the age of 18; 20% had one such child; 19% had two children; 7% had three children; and 3% had four or more children. If a family is selected at random, find the probability that the family has:

(a) Two or three children.
(b) More than one child.
(c) Less than four children.

(a) .26  (b) .29  (c) .97

**Example 7:** Suppose we are drawing a card from a standard deck of cards. What are the following probabilities?

(a) Getting a 5 or a jack.
(b) Getting a heart or spade.
(c) Getting a 10 or a diamond.

(a) There are 8 cards that are 5’s or jacks, so the probability is

\[ P(5 \text{ or } J) = \frac{8}{52} = \frac{2}{13}. \]

(b) There are 26 such cards, so \( P(H \text{ or } S) = \frac{26}{52} = \frac{1}{2}. \)

(c) There are 16 such cards, so \( P(10 \text{ or } D) = \frac{16}{52} = \frac{4}{13}. \)

Are parts (a) and (b), we just added the two individual probabilities together, but on part (c) that didn’t work. Why not?

**Definition:** Two events are **mutually exclusive** if they cannot occur at the same time.
Addition Rule For Probabilities

1. If $A$ and $B$ are mutually exclusive events, then

$$P(A \text{ or } B) = P(A) + P(B).$$

2. If $A$ and $B$ are not mutually exclusive events, then

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B).$$

Example 8: In a statistics class with 24 students, there are 11 seniors and 13 juniors. Nine of the seniors are female and 7 of the juniors are male. If a student is selected at random, find the probability that the selected student is: (a) a junior or a female, (b) a senior or a female, (c) a junior or a senior.

(a) $P(J \text{ or } F) = P(J) + P(F) - P(J \text{ and } F)$

$$= \frac{13}{24} + \frac{15}{24} - \frac{6}{24}$$

$$= \frac{22}{24} = \frac{11}{12}$$

(b) $P(S \text{ or } F) = P(S) + P(F) - P(S \text{ and } F)$

$$= \frac{11}{24} + \frac{15}{24} - \frac{9}{24}$$

$$= \frac{17}{24}$$

(c) $P(J \text{ or } S) = P(J) + P(S)$

$$= \frac{13}{24} + \frac{11}{24}$$

$$= \frac{24}{24} = 1$$