Section 2: Algebraic Numbers

Diophantine equations are very special equations in that we only wish to find integral solutions. Not all equations have this characteristic. In fact, for most equations, we are interested in finding all of the solutions. In this section, we will no longer restrict ourselves to only integer solutions. We will now allow rational and irrational number solutions, and in so doing, study arithmetic and algebraic numbers. As I often do, we begin with some history.

The rational numbers are historically a very well accepted set of numbers. The integers of course have been part of our mathematical consciousness since man began counting (as much as 30,000 years ago). Since rational numbers are just fractions of integers, they too have been embraced for thousands of years. The irrational numbers are another story. A real number is said to be ***irrational*** if it is not rational. Stated more directly, a number is irrational if its decimal expansion is non-terminating and non-repeating. This definition is not much better, for how do we prove that a decimal expansion *never* ends or repeats. Maybe we just haven't carried out enough digits. Indeed the very existence of such numbers was questionable to some until very recently (the 1800's). Clearly now they are accepted and in fact they make up a large portion of the real numbers.

Definition 3.2.1 A number x is ***arithmetic*** (the accent is on the third syllable, like the word "algebraic") if it can be expressed using only natural numbers and the operations of addition, subtraction, multiplication, division, exponentiation, and root taking.

Example 3.2.2 The following numbers are all arithmetic:

$$100, \quad -\frac{2}{17}, \quad \sqrt{2}, \quad \sqrt[3]{13}, \quad \sqrt{3}+\sqrt[4]{5}, \quad \frac{25-\sqrt[31]{201}}{41\sqrt{5}}.$$

Clearly the sets of natural numbers, integers, and rational numbers are all arithmetic. But these numbers are just the beginning. All of the listed numbers are real numbers also. Does this mean that every arithmetic number is real? Nope. Clearly, we can write $i=\sqrt{-1}=\sqrt{(2-2)-1}$ and see that i is also arithmetic. We will deal more with complex numbers in Section 3.3. Some arithmetic numbers appear to be irrational, but this can be deceiving. The number $\sqrt{2}/\sqrt{8}$ looks irrational, but in fact is the rational number $1/2$. So how do we prove which arithmetic numbers are irrational?

Theorem 3.2.3 Let m be a natural number and suppose that $m \neq k^n$ for any natural number k. Then $\sqrt[n]{m}$ is irrational.

Proof: Suppose to the contrary that $\sqrt[n]{m}=\dfrac{r}{s}$ for some integers r and s, and further suppose that the fraction $\frac{r}{s}$ is in lowest terms (i.e. $\gcd(r,s)=1$). Then $m=\left(\frac{r}{s}\right)^n$, and so we get that $ms^n=r^n$. It follows that $m \mid r^n$ and since $\gcd(r,s)=1$, we also have that $r^n \mid m$. Therefore $m=r^n$, which is a contradiction to the hypothesis of the theorem. Therefore, $\sqrt[n]{m}$ is irrational. ■

Exercise 3.2.4: Prove or disprove: If a is an integer and $\sqrt[n]{a}$ is rational, then $\sqrt[n]{a}$ is an integer.

Theorem 3.2.5 Let $N=r+s\sqrt[n]{m}$. If $\sqrt[n]{m}$ is irrational and r and s are rational, then N is irrational.

Proof: Notice that $\sqrt[n]{m}=\dfrac{N-r}{s}$. Therefore, if N is rational, so is $\sqrt[n]{m}$ (which we know is false by assumption). So N is irrational. ■

These theorems provide us with many examples of irrational numbers. But what can we say about sums, products, and powers of irrational numbers?

Example 3.2.6 Is the number $N=\sqrt{2}+\sqrt{3}$ rational or irrational? If we progress as follows,

$$\begin{aligned} N &= \sqrt{2}+\sqrt{3} \\ N-\sqrt{2} &= \sqrt{3} \\ \left(N-\sqrt{2}\right)^2 &= 3 \\ N^2-2N\sqrt{2}+2 &= 3 \\ N^2-1 &= 2N\sqrt{2} \\ \frac{N^2-1}{2N} &= \sqrt{2} \end{aligned}$$

we see that if N is rational, then so is $\sqrt{2}$, which we know is false. Therefore $N=\sqrt{2}+\sqrt{3}$ is irrational.

Exercise 3.2.7 Show that $N=\sqrt[3]{2}+\sqrt[3]{3}$ is irrational.

Now we will focus our attention back to equations and their solutions.

Definition 3.2.8 A number x is ***algebraic*** if it is a solution to a polynomial equation with integer coefficients.

Notice that we could have replaced the word "integer" in this definition with "rational." Any polynomial equation with rational coefficients could be multiplied by the least common denominator and the resulting equation would then have integer coefficients. As was the case with arithmetic numbers, the sets of natural numbers, integers, and rational numbers are all algebraic. The roots of these numbers are algebraic also.

Example 3.2.9 The number i is clearly algebraic since it is a solution to the equation $x^2+1=0$.

Example 3.2.10 $N=\sqrt{2}+\sqrt{3}$ is algebraic.

To show this, we need to find a polynomial equation with integer coefficients that it satisfies. Recall that during the example where we showed that it was arithmetic, we arrived at the equation $N^2-1=2N\sqrt{2}$. If we square both sides of this equation, we get $\left(N^2-1\right)^2=8N^2$, which simplifies to $N^4-10N^2+1=0$. So N is a solution to the polynomial equation $x^4-10x^2+1=0$.

It is natural to wonder what the other three roots of this polynomial are. They are $\sqrt{2}-\sqrt{3}$, $-\sqrt{2}+\sqrt{3}$, and $-\sqrt{2}-\sqrt{3}$. Therefore, these are all algebraic numbers as well.

Exercise 3.2.11 Show that $N=\sqrt{2}+\sqrt[4]{2}$ is algebraic.

Definition 3.2.12 A number x is ***transcendental*** is it is not algebraic.

There are no guarantees that any such number actually exists. Perhaps all real numbers are algebraic. Not very likely is it? It was not until the mid-1800's that the term transcendental was even coined. Prior to this no one believed that such numbers existed. Keep in mind that it took until the mid-1700's to prove that such famous numbers such as π and e were even irrational. Showing that they (or any other candidates) were not algebraic took another 100 years. In 1844, Joseph Liouville of France proved that the number

$$0.110001000000000000000001000...$$

(with a one in the digits corresponding to $10^{-(n!)}$ for each n and zeros elsewhere) was transcendental. Then in 1872, the French mathematician Charles Hermite showed that e was transcendental and in 1882, the German Ferdinand Lindemann proved the transcendence of π. Even so, transcendental numbers seemed to be quite rare.

Then the most amazing thing happened. Russian mathematician Georg Cantor showed that *almost all* real numbers are transcendental. He did this by using his set theory notions that he was developing at the time. Cantor was investigating infinite sets. He defined a set as countable if it could be put in one-to-one correspondence with the set of natural numbers. Such a set was called *countably infinite*. Then he showed that the set of real numbers was *uncountably infinite*. Therefore it was significantly larger than the set of natural numbers, even though both were infinite. The big news with regard to transcendental numbers was this: Cantor showed that the set of algebraic numbers was countable. Since every real number was either algebraic or not algebraic (transcendental), and there were uncountably many real numbers, there must be uncountably many transcendental numbers as well. Amazing. As rare as they seemed, they were everywhere.

In 1900, David Hilbert posed 23 mathematical problems that he believed should be at the forefront of research in the upcoming years. Problem number 7 asked whether the number $2^{\sqrt{2}}$ was algebraic or transcendental. In 1934, Russian mathematician Aleksander Gelfond showed that $2^{\sqrt{2}}$ was transcendental and in fact he showed a whole lot more. He found a whole class of numbers that were transcendental.

Theorem 3.2.13 (Gelfond's Theorem) Let x and y be two numbers (not necessarily real). The number x^y is transcendental if x is algebraic (but not 0 or 1) and y is irrational and algebraic.

(Notice that $2^{\sqrt{2}}$ fits the hypothesis of this theorem.)

Exercise 3.2.14 Show that e^{π} is transcendental. (Hint: You may want to use Euler's beautiful formula $e^{\pi i}+1=0$. As an aside, isn't it amazing that those 5 important mathematical quantities are all related so simply? Wow.)

Example 3.2.15 Let's consider a few different powers of 10.

(a) 10^{-1}. This number is rational and algebraic.
(b) $10^{1/2}$. This number is irrational and algebraic.
(c) $10^{\sqrt{2}}$. This number is transcendental by Gelfond's Theorem.
(d) $10^{\log 2}$. This is the number 2, and is therefore clearly an integer and algebraic.

Theorem 3.2.16 If x is a natural number that is not a power of 10, then $\log x$ is transcendental.

Proof: Let $y=\log x$ and suppose that y is algebraic. Since x is not a power of 10, y is irrational. Therefore, 10^y is transcendental (by Gelfond's Theorem). But this is a contradiction, since $10^y = x$. So, $y=\log x$ is transcendental. ■