CHAPTER IV: GRAPH THEORY

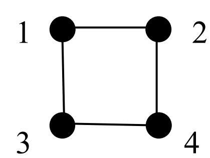
Section 1: Introduction to Graphs

Since this class is called "Number-Theoretic and Discrete Structures", it would be a crime to only focus on number theory – regardless how wonderful those topics are. In this section, we will introduce a popular object of study in discrete mathematics that have an interesting application to number theory. I am a number theorist after all.

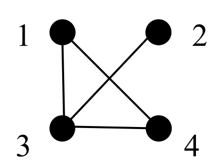
Definition 4.1.1 A ***graph*** G is a pair of sets in which every element of the second set is an unordered pair of elements from the first. The elements of the first set are called the ***vertices*** of the graph and the elements of the second set are called the ***edges***. The vertex set will be denoted by $V(G)$ and the edge set by $E(G)$. Usually, the vertex set is not allowed to be empty or infinite (although infinite graphs do exist) however the edge set is can be empty.

Example 4.1.2 The pair of sets $V(G)=\{1,2,3,4\}$ and $E(G)=\{\{1,2\},\{1,3\},\{2,4\},\{3,4\}\}$ forms a graph G. For simplicity, we will denote the edge set as $E(G)=\{12,13,24,34\}$.

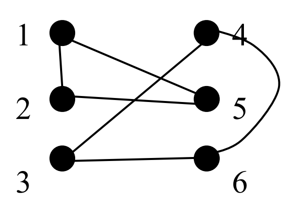
Graphs are most often studied by drawing them graphically. The vertices are represented by dots and the edges are line segments (or arcs) connecting the correct vertices. For example, the graph of Example 4.1.2 would look like:



Example 4.1.3 Let graph G have $V(G)=\{1,2,3,4\}$ and $E(G)=\{14,13,23,34\}$.

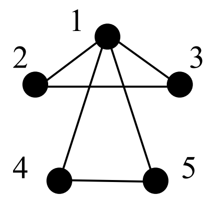


Example 4.1.4 Let graph G have $V(G)=\{1,2,3,4,5,6\}$ and $E(G)=\{12,15,25,34,36,46\}$.



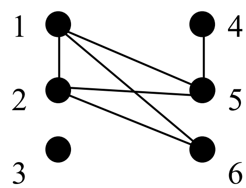
Notice that since the edges are denoted by unordered pairs, there are two ways to denote each edge. The edge connecting vertices 1 and 4 in the above graph could be 14 or 41. I will usually default to writing the lower number first.

Example 4.1.5 What are the vertex and edge sets for the graph represented below?



$V(G) = \{1,2,3,4,5\}$ and $E(G) = \{12,13,14,15,23,45\}$.

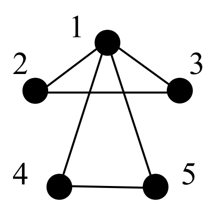
Exercise 4.1.6 What are the vertex and edge sets for the graph represented below?



Definition 4.1.7 A graph is ***connected*** if there is a path of edges between every pair of vertices. Otherwise it is called ***disconnected***.

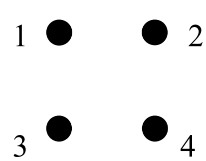
The graphs depicted in Examples 4.1.3 and 4.1.6 are connected. Example 4.1.4 is disconnected. There is no path from vertex 1 to vertex 3 (for example).

Exercise 4.1.8 How many distinct paths can you find between vertices 1 and 2? Each path must start at 1 and end at 2. It can repeat vertices, but not edges.

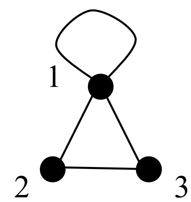


As alluded to above, there are some special cases to consider.

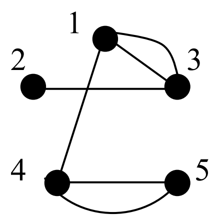
Example 4.1.9 Let $V(G) = \{1,2,3,4\}$ and $E(G) = \varnothing$. Graphs of this type are called ***totally disconnected graphs.***



Example 4.1.10 Let $V(G)=\{1,2,3\}$ and $E(G)=\{11,12,13,23\}$. The edge 11 is called a ***loop***.



Example 4.1.11 Let $V(G)=\{1,2,3,4,5\}$ and $E(G)=\{13,13,14,23,45,45\}$. Here we see ***multiple edges***.

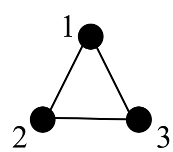


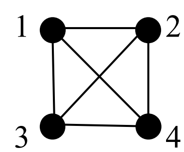
Definition 4.1.12 A ***simple*** graph is a graph with no loops or multiple edges. If two vertices are connected by an edge, they are said to be ***adjacent*** to one another or ***neighbors***. The ***degree*** of a vertex (in a simple graph) is its number of neighbors. For the vertex v, its degree is denoted by $d(v)$. [Note: Generalizing the notion of degree to graphs with loops and multiple edges can be done, but we will generally restrict ourselves to simple graphs.] A ***complete graph*** is one in which every pair of distinct vertices are connected by an edge. The complete graph with n vertices is denoted by K_n.

Example 4.1.13 Consider the simple graph from Example 4.1.3. Find the degree of each vertex.

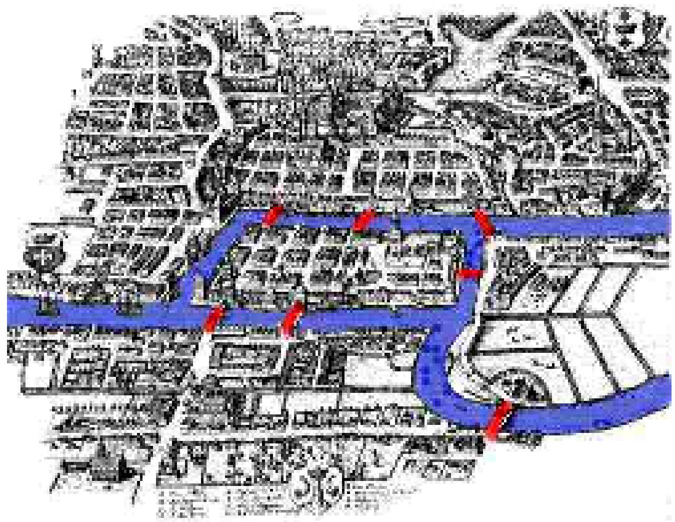
$d(1)=2$, $d(2)=1$, $d(3)=3$, $d(4)=2$

Example 4.1.14 K_3 and K_4

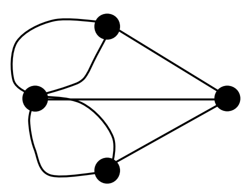




It might be time for a little history. It is often difficult to pinpoint the birth of a field of mathematics. With graph theory, that's not the case. The birth of graph theory is generally considered to be around 1736, when Leonhard Euler solved a longstanding problem called the *Königsberg Bridge Problem*. Königsberg was a town in Prussia located on the Pregel river. The city occupied two islands in the river plus areas on both banks. These regions were connected by seven bridges (see picture).



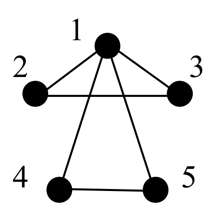
The residents wondered whether they could leave their home, take a long walk through town in which they visited every bridge exactly once, and then return to their home. If we represent this town by a graph (the vertices are the 4 regions of town and the edges are the 7 bridges), we get something like the following graph:



Euler was able to show that no such path existed. Paths of that kind are now called ***Eulerian circuits*** and graph that have them are called ***Eulerian graphs***.

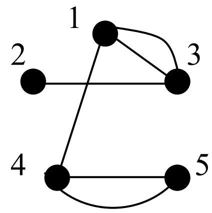
Definition 4.1.15 A graph is ***Eulerian*** is there exists a path beginning and ending at the same vertex which uses every edge once and only once.

Example 4.1.16 Consider the graph from Example 4.1.8:

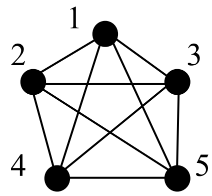


Starting with vertex 1, we can find an Eulerian circuit as follows: 1,3,2,1,5,4. Notice this path begins and ends at the same vertex and uses every edge exactly once.

Example 4.1.17 The graph from Example 4.1.11 however, has no Eulerian circuit. In any Eulerian circuit, there must be at least two edges at every vertex; one to enter and one to leave. But the degree of vertex 2 is 1.

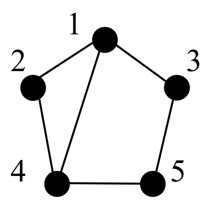


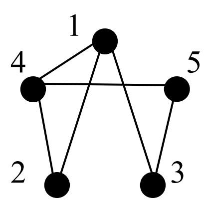
Exercise 4.1.18 Find an Eulerian circuit in the given graph or explain how you know there isn't one. You may write your circuit like I did in Example 4.1.16.

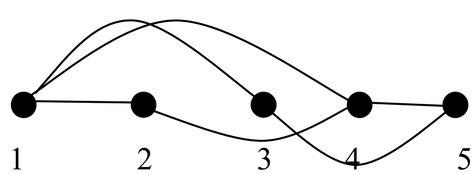


Theorem 4.1.19 A connected graph with at least one vertex is Eulerian if and only if every vertex has even degree.

Example 4.1.20 Note that there are many ways to represent a graph. What the graph looks like is not important, just the relationships between vertices it represents. Consider $V(G)=\{1,2,3,4,5\}$ and $E(G)=\{12,13,14,24,45,35\}$. Each of the visual representations below is accurate.





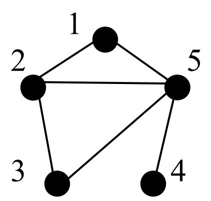


Definition 4.1.21 A ***vertex decomposition*** of a graph G is an unordered pair of subsets $\{U_1, U_2\}$ of the vertex set of G with

(1) $U_1 \cup U_2 = V(G)$, and
(2) $U_1 \cap U_2 = \varnothing$.

Since the pair of subsets form a partition of $V(G)$, we actually only need to explicitly define one of the subsets and the other would be "everything else." A vertex is ***special with respect to the vertex decomposition*** $\{U, V(G) - U\}$ if it is adjacent to an odd number of vertices *in the subset to which it does not belong*. We denote the set of special vertices by $P(G, U)$.

Example 4.1.22 Consider the graph G:



If we let $U = \{1, 2\}$, then we have $V(G) - U = \{3, 4, 5\}$. We can see that the vertex 1 is adjacent to only one vertex in $V(G) - U$ and the vertex 3 is adjacent to only one vertex in U. All other vertices are adjacent to an even number of vertices in the subset to which they do not belong. Therefore, for this vertex decomposition, $P(G, U) = \{1, 3\}$. If we had chosen a different vertex decomposition, say $U = \{1, 3\}$, we would possibly get a different set of special vertices (in this case $P(G, U) = \varnothing$).

Theorem 4.1.23

(a) For any graph G, $P(G, U) = P(G, V(G) - U)$.
(b) $P(G, \varnothing) = \varnothing$.
(c) If G is a totally disconnected graph, then $P(G, U) = \varnothing$ for any $U \subseteq V(G)$.
(d) If $G = K_n$ (with n even), then for any subset $U \subseteq V(G)$ with an odd number of vertices, $P(K_n, U) = V(K_n)$.
(e) If $G = K_n$ (with n odd), then for any nonempty proper subset $U \subseteq V(G)$ (i.e. $U \neq \varnothing$ and $U \neq V(K_n)$), $P(K_n, U) \neq \varnothing$.

Several parts of this theorem are quite trivial (how can any vertex be special with respect to $U = \varnothing$?), but you're asked to prove them in the last exercise.

Exercise 4.1.24 Prove Theorem 4.1.23.