CHAPTER I: Vector Spaces

Section 1: Introduction and Examples

This first chapter is largely a review of topics you probably saw in your linear algebra course. So why cover it?

(1) Not everyone remembers everything they have previously seen.
(2) Not every linear algebra course is the same, so it's possible some people never saw this material.
(3) Even if you have already seen it, and still remember it, it's very important, so twice is nice.

In mathematics, we study many different objects. We usually begin with definitions of the objects to be studied and then maybe introduce a few of their basic characteristics. Very quickly however, the question turns to "What can we do with these?" In other words, given two of these things, how can I combine them or change them and produce new objects. That is where operations come in. In some cases, operations work on single objects (such as differentiation) and in others they require a pair of objects (such as addition or multiplication). In linear algebra, we have another situation. We have sets of objects that have two operations defined on them: one that pairs the objects up (addition) and another that pairs objects with other numbers, called scalars (scalar multiplication). If the operations satisfy a certain list of axioms, the set is called a vector space.

The term "vector space" is a very good one, but also can be misleading. Normal Euclidean space $\mathbb{R}^2$ is indeed a vector space, and its elements often called vectors, but in more general vector spaces, we have to be very careful to understand which objects are the vectors and which are the scalars. This is not difficult, but just requires attention. So what are the axioms that the operations on a set of vectors must satisfy? I'm glad you asked.

Definition 1.1.1 Let V be a set closed under two operations, usually denoted by (vector) addition and scalar multiplication. The set is called a ***vector space*** if the following axioms are satisfied:

(1) $x+y=y+x$ for all $x,y\in V$.
(2) $(x+y)+z=x+(y+z)$ for all $x,y,z\in V$.
(3) there exists an element of V, usually denoted by 0, such that $x+0=x$ for all $x\in V$. (This element is called the ***zero vector*** or the ***identity***.)
(4) for every $x\in V$ there exists an element, usually denoted by $-x\in V$, such that $x+(-x)=0$. The element $-x$ is called the ***inverse*** of x.
(5) $\alpha(x+y)=\alpha x+\alpha y$ for all $x,y\in V$ and scalar α.
(6) $(\alpha+\beta)x=\alpha x+\beta x$ for all $x\in V$ and scalars α,β.
(7) $(\alpha\beta)x=\alpha(\beta x)$ for all $x\in V$ and scalars α,β.
(8) $1x=x$ for all $x\in V$.

Note: The set of scalars will usually be assumed to be the real numbers unless otherwise stated. To be clear, we will often indicate the set of scalars by saying "V is a ***vector space over*** $\mathbb{R}$" or "V is a ***vector space over*** $\mathbb{C}$". (For those with some exposure to modern algebra: In general, the set of scalars can come from any field F. In this case we say that V is a ***vector space over*** F. If the set of scalars is only a ring, then the set of "vectors" is called a ***module***.)

Example 1.1.2 $\mathbb{R}^2$

$\mathbb{R}^2$ is ordinary Euclidean space; i.e. the set of all two dimensional vectors with real entries. Vector addition is defined component-wise, as is multiplication by real numbers. It is a straightforward procedure to verify the eight vector space axioms and see that $\mathbb{R}^2$ is a vector space over $\mathbb{R}$.

Example 1.1.3 Similarly, $\mathbb{R}^n$ (the set of all n-dimensional vectors with real entries) is a vector space over $\mathbb{R}$ for any natural number n. In particular, if $n=1$ we see that in fact $\mathbb{R}$ is a vector space over itself.

Example 1.1.4 $\mathbb{R}^{m\times n}$

This is the set of $m\times n$ matrices with real entries. You might recall that under ordinary matrix addition (which is also component-wise) and scalar multiplication, all of the above properties are satisfied.

Example 1.1.5 Let P_n denote the set of all polynomials of degree less than n. We will add polynomials and multiply by real numbers as usual. Notice that the sum of two polynomials of degree less than n also has a degree less than n, so this set is closed under addition. Similarly, multiplying a polynomial by a scalar can only reduce its degree. The zero polynomial is the identity. For $p \in P_n$, its inverse is $-p$.

Example 1.1.6 Let $C[a,b]$ denote the set of all continuous real-valued functions that are defined on $[a,b]$. Our two operations are again addition and scalar multiplication. Since the sum of two continuous functions is still continuous (and the same hold for scalar multiples), this set is closed under these operations. The zero function is continuous and is therefore the identity. For $f \in C[a,b]$, its inverse is $-f$.

So far, all of the examples have used ordinary operations and objects, so I haven't bothered to verify the axioms in great detail. Let's get a few more interesting examples.

Example 1.1.7 Let S be the set of all ordered pairs of real numbers. (Note that I'm not calling this $\mathbb{R}^2$. That symbol is reserved for Euclidean space with normal addition and scalar multiplication.) Define scalar multiplication as normal

$$\alpha(x_1, x_2) = (\alpha x_1, \alpha x_2),$$

but define vector addition by

$$(x_1, x_2) \oplus (y_1, y_2) = (x_1 + y_1, 0).$$

(Another note: I'm using the $\oplus$ symbol to distinguish it from normal addition.)

Let's check the axioms and see if this is a vector space.

A1: $(x_1, x_2) \oplus (y_1, y_2) = (x_1 + y_1, 0) = (y_1 + x_1, 0) = (y_1, y_2) \oplus (x_1, x_2)$. ☑
(One more note: To verify that $\oplus$ was commutative, I was able to use the fact that ordinary addition + is already commutative. We will use this technique again for associativity on A2.)

A2:

$$\begin{aligned}(x_1, x_2) \oplus \big((y_1, y_2) \oplus (z_1, z_2)\big) &= (x_1, x_2) \oplus (y_1 + z_1, 0) \\ &= (x_1 + (y_1 + z_1), 0) \\ &= ((x_1 + y_1) + z_1, 0) \\ &= (x_1 + y_1, 0) \oplus (z_1, z_2) \\ &= \big((x_1, x_2) \oplus (y_1, y_2)\big) \oplus (z_1, z_2) \quad ☑\end{aligned}$$

A3: This axiom fails. There is no zero element. Let $(x_1, x_2) \in S$ with $x_2 \neq 0$. Adding any element to (x_1, x_2) will yield an element whose second coordinate is 0. Note: It would <u>not</u> have been enough to just show that $(0,0)$ failed to be the zero element. It is possible in abstract vector spaces for some unexpected element to play the role of 0.

Since one axiom fails, S is not a vector space. We do not need to check the others.

Example 1.1.8 Let $\mathbb{R}^+$ denote the set of positive real numbers. Define addition by $x \oplus y = xy$, and scalar multiplication by $\alpha \circ x = x^\alpha$.

A1: $x \oplus y = xy = yx = y \oplus x$.☑
A2: $x \oplus (y \oplus z) = x \oplus yz = x(yz) = (xy)z = (x \oplus y) \oplus z$.☑
A3: (Here is an example of a counterintuitive zero element. Just because the number 0 doesn't work, doesn't mean A3 fails.)
$x \oplus 1 = x1 = x$.☑
A4: (Note that the target product here is the zero element, not the number 0.) Let $x \in \mathbb{R}^+$. Since x is not 0, $\frac{1}{x} \in \mathbb{R}^+$ also. Then we have $x \oplus \frac{1}{x} = x \cdot \frac{1}{x} = 1$.☑

Axioms 5 and 6 are a good exercise in working with notation. Using the operations of this example, Axiom 5 states $\alpha \circ (x \oplus y) = \alpha \circ x \oplus \alpha \circ y$ and Axiom 6 states $(\alpha + \beta) \circ x = \alpha \circ x \oplus \beta \circ x$. That is what we need to verify.

A5: $\alpha \circ (x \oplus y) = \alpha \circ (xy) = (xy)^{\alpha} = x^{\alpha} y^{\alpha} = x^{\alpha} \oplus y^{\alpha} = \alpha \circ x \oplus \alpha \circ y$.☑
A6: $(\alpha + \beta) \circ x = x^{\alpha+\beta} = x^{\alpha} x^{\beta} = x^{\alpha} \oplus x^{\beta} = \alpha \circ x \oplus \beta \circ x$.☑
A7: $(\alpha\beta) \circ x = x^{\alpha\beta} = x^{\beta\alpha} = (x^{\beta})^{\alpha} = \alpha \circ (x^{\beta}) = \alpha \circ (\beta \circ x)$.☑
A8: $1 \circ x = x^{1} = x$.☑

So $\mathbb{R}^{+}$ under these operations is a vector space.

Exercise 1.1.9 Let $\mathbb{Z}$ be the set of integers. Define addition in the usual way but define scalar multiplication by $\alpha \circ k = [\![\alpha]\!] \cdot k$. The dot $\cdot$ represents normal multiplication and the double brackets $[\![\ \ ]\!]$ represent the greatest integer function. Is this a vector space over $\mathbb{R}$? If yes, verify all 8 axioms. If no, find one axiom that fails.

Exercise 1.1.10 Let S be the set of all ordered pairs of real numbers. Define $(x_1, x_2) \oplus (y_1, y_2) = (x_1 + y_1 + 1, x_2 + y_2 + 1)$ and $\alpha \circ (x_1, x_2) = (\alpha + \alpha x_1 - 1, \alpha + \alpha x_2 - 1)$. Is this a vector space over $\mathbb{R}$? If yes, verify all 8 axioms. If no, find one axiom that fails.

Exercise 1.1.11 Let $\mathbb{R}$ be the set of real numbers with scalar addition defined as usual, but addition defined by $x \oplus y = \max\{x, y\}$. Is this a vector space over $\mathbb{R}$? If yes, verify all 8 axioms. If no, find one axiom that fails.

The eight axioms required of vector spaces are not the only properties that are true. How is an axiom different from a property or characteristic? Axioms are properties that cannot be deduced from other properties, so they have to be verified up front. All other properties about vector spaces can be deduced from these eight. For instance…

Example 1.1.12 In every vector space V, $0x = 0$ for all $x \in V$.

(Here I am using the symbol "0" in two different ways in one equation. The first "0" is the scalar 0. The "0" on the right hand side is the zero element, which as we have seen may or may not be the number 0. Confused yet?)

First of all, note that $x + 0x = 1x + 0x = (1+0)x = 1x = x$. Adding $-x$ to the far left and far right of that equation gives us $0x = 0$.

Example 1.1.13 In every vector space V, $\alpha 0 = 0$ for all scalar α.

(Note: Both "0"'s are the zero element.) First note that $\alpha 0 + \alpha 0 = \alpha(0+0) = \alpha 0$. Now add $-(\alpha 0)$, the inverse of $\alpha 0$, to both sides.

$$\begin{aligned}\left(\alpha 0+\alpha 0\right)+-(\alpha 0) &= \alpha 0+-(\alpha 0)\\ \alpha 0+\left(\alpha 0+-(\alpha 0)\right) &= 0\\ \alpha 0+0 &= 0\\ \alpha 0 &= 0\end{aligned}$$

Exercise 1.1.14 Prove that $-1x=-x$ for all $x\in V$.

Exercise 1.1.15 Prove that if $\alpha x=0$, then either $\alpha=0$ or $x=0$.

Exercise 1.1.16 Prove that $x+y=0$ implies $y=-x$ (i.e. inverses are unique).