CHAPTER I: Vector Spaces

Section 2: Subspaces

Definition 1.2.1 If S is a non-empty subset of a vector space V and S is also closed under the two operations of V, then S is a ***subspace*** of V. Every non-trivial vector space V has at least two subspaces; namely $\{0\}$ and V. Any other subspace is called a ***proper subspace***.

Example 1.2.2 Let $X = \{(x_1, x_2)^T : x_1 = 3x_2\}$. This is a subset of $\mathbb{R}^2$ (with the usual operations). Technically, we consider the elements of $\mathbb{R}^2$ column vectors $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, but it is more convenient to write them using their transpose $(x_1, x_2)^T$. Is X a subspace of $\mathbb{R}^2$?

We just need to check closure. Let $(3x, x)^T$ and $(3y, y)^T$ be two arbitrary elements of X. Then $(3x, x)^T + (3y, y)^T = (3x + 3y, x + y)^T = (3(x + y), x + y)^T$ and $\alpha(3x, x)^T = (\alpha(3x), \alpha x)^T = (3(\alpha x), \alpha x)^T$ are clearly still in X.

Example 1.2.3 $X = \{(x_1, x_2)^T : x_2 = 2x_1 + 1\}$. This is not a subspace of $\mathbb{R}^2$ since it is not closed under vector addition:

$$\begin{aligned} (x, 2x+1)^T + (y, 2y+1)^T &= (x + y, 2x + 1 + 2y + 1)^T \\ &= (x + y, 2(x + y) + 2)^T \\ &\neq (x + y, 2(x + y) + 1)^T \end{aligned}$$

(In fact, it's not closed under scalar multiplication either.)

Example 1.2.4 P_n is a proper subspace of $C[\mathbb{R}]$ for any n.

Clearly $P_n \neq \varnothing$ for any n. Since polynomials are continuous and defined everywhere, $P_n \subseteq C[\mathbb{R}]$. We already have seen that P_n is closed under function addition and scalar multiplication. To see that P_n is a *proper* subspace, just note that there exist continuous functions on $\mathbb{R}$ that are not polynomials (e.g. $f(x) = \cos(x)$). So $P_n \neq C[\mathbb{R}]$.

Exercise 1.2.5 Which of the following are subspaces of $\mathbb{R}^2$? Explain.

(a) $\{(x_1, x_2)^T : |x_1| = |x_2|\}$

(b) $\{(x_1, x_2)^T : x_1 + x_2 = 0\}$

(c) $\{(x_1, x_2)^T : x_1 x_2 = 0\}$

(d) $\{(x_1, x_2)^T : x_1 = x_2\}$

Exercise 1.2.6 Which of the following are subspaces of $\mathbb{R}^{2\times 2}$? Explain.

(a) all 2×2 symmetric matrices. (b) all 2×2 matrices A for which $a_{12}=1$.

(c) all 2×2 singular matrices. (d) all 2×2 matrices B for which $b_{11}=0$.

Exercise 1.2.7 Which of the following are subspaces of P_4?

(a) all polynomials in P_4 of even degree. (b) all polynomials of degree 3.

(c) all $p(x)$ in P_4 for which $p(0)=0$. (d) all constant polynomials.

Definition 1.2.8 Let A be an $m\times n$ matrix. The ***nullspace*** of A is the set

$$N(A)=\{x\in\mathbb{R}^n : Ax=0\}.$$

Example 1.2.9 Find the nullpspace of $A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$.

If we rewrite the matrix equation $\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$ as a system, we get

$$\begin{aligned}x_1+2x_2&=0\\ 3x_1+4x_2&=0\end{aligned}.$$

Solving this system, we see that $x_1=0$, $x_2=0$ is the only solution. So $N(A)=\{0\}$.

Exercise 1.2.10 Find the nullspace of $A=\begin{bmatrix}3 & 1\\ -2 & 2\end{bmatrix}$.

Question: Clearly $N(A)$ is a subset of $\mathbb{R}^n$. Is it a *subspace*?
Answer: If not, we named it poorly.

Theorem 1.2.11 Let $A\in\mathbb{R}^{m\times n}$. The $N(A)$ is a subspace of $\mathbb{R}^n$.

Exercise 1.2.12 Prove Theorem 1.2.8.

Definition 1.2.13 Let V be a vector space and let $v_1,v_2,\ldots,v_n\in V$. A sum of the form $a_1v_1+a_2v_2+\ldots+a_nv_n$ (where a_i are scalars) is called a ***linear combination*** of the vectors $v_1,v_2,\ldots,v_n$. The set of all linear combinations of a set of vectors is the ***span*** of the vectors. This is denoted by $\operatorname{Span}(v_1,v_2,\ldots,v_n)$.

Theorem 1.2.14 Let V be a vector space and let $v_1,v_2,\ldots,v_n\in V$. Then the $\operatorname{Span}(v_1,v_2,\ldots,v_n)$ is a subspace of V.

Proof: Let $x, y \in \text{Span}(v_1, v_2, \ldots, v_n)$ and $\alpha \in \mathbb{R}$. So there exist $a_i \in \mathbb{R}$ and $b_i \in \mathbb{R}$ such that $x = a_1 v_1 + a_2 v_2 + \ldots + a_n v_n$ and $y = b_1 v_1 + b_2 v_2 + \ldots + b_n v_n$. Then

$$\begin{aligned} x + y &= (a_1 v_1 + a_2 v_2 + \ldots + a_n v_n) + (b_1 v_1 + b_2 v_2 + \ldots + b_n v_n) \\ &= (a_1 + b_1) v_1 + (a_2 + b_2) v_2 + \ldots + (a_n + b_n) v_n \end{aligned}, \text{ and}$$

$$\begin{aligned} \alpha x &= \alpha(a_1 v_1 + a_2 v_2 + \ldots + a_n v_n) \\ &= (\alpha a_1) v_1 + (\alpha a_2) v_2 + \ldots + (\alpha a_n) v_n \end{aligned}.$$

So $x + y \in \text{Span}(v_1, v_2, \ldots, v_n)$ and $\alpha x \in \text{Span}(v_1, v_2, \ldots, v_n)$.

So $\text{Span}(v_1, v_2, \ldots, v_n) \subseteq V$. If the $\text{Span}(v_1, v_2, \ldots, v_n) = V$, then there seems to be something special about $\{v_1, v_2, \ldots, v_n\}$. Indeed.

Definition 1.2.15 If $\text{Span}(v_1, v_2, \ldots, v_n) = V$, then the set $\{v_1, v_2, \ldots, v_n\}$ is a ***spanning set*** of V. In other words, the set $\{v_1, v_2, \ldots, v_n\}$ is a spanning set of V if and only if every vector in V is a linear combination of the vectors $v_1, v_2, \ldots, v_n$.

Example 1.2.16 Are the following sets spanning sets for $\mathbb{R}^2$?

(a) $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$ (b) $\left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \end{bmatrix} \right\}$

(a) Can every vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$ be written as a linear combination of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$? We need to find the coefficients (if they exist) for which

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = a_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + a_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

In other words, we need to solve the system

$$\begin{aligned} a_1 + 3a_2 &= x_1 \\ 2a_1 + a_2 &= x_2 \end{aligned}.$$

Solving for a_1 and a_2, we get $a_1 = -\frac{1}{5} x_1 + \frac{3}{5} x_2$ and $a_2 = \frac{2}{5} x_1 - \frac{1}{5} x_2$. So “Yes.”

(b) Can every vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$ be written as a linear combination of $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 6 \end{bmatrix}$? We need to find the coefficients (if they exist) for which

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = a_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + a_2 \begin{bmatrix} 4 \\ 6 \end{bmatrix}.$$

In other words, we need to solve the system

$$\begin{aligned} 2a_1 + 4a_2 &= x_1 \\ 3a_1 + 6a_2 &= x_2 \end{aligned}.$$

However, this system has no solutions if $x_2 \neq \frac{3}{2} x_1$. So "No."

Exercise 1.2.17 Which of the following are spanning sets for $\mathbb{R}^3$?

(a) $\{(1,0,0)^T, (0,1,1)^T, (1,0,1)^T\}$

(b) $\{(1,0,0)^T, (0,1,1)^T, (1,0,1)^T, (1,2,3)^T\}$

(c) $\{(2,1,-2)^T, (3,2,-2)^T, (2,2,0)^T\}$

Exercise 1.2.18 We can think of $\mathbb{R}$ as a vector space over itself. So both the vectors and the scalars are real numbers. With this in mind, prove that there are no proper subspaces of $\mathbb{R}$.