Section 3: Basis and Dimension

Recall Exercise 1.2.17. We saw that the collection of vectors in part (a) was a spanning set for $\mathbb{R}^3$, as was the collection of vectors in part (b), since it included all the vectors from part (a) as a subset. That stands to reason in general; if you add a vector (or vectors) to a spanning set, you will still have a spanning set. However, this would create sort of "trivial" spanning sets. What we desire is a "minimal" spanning set for a vector space. This is our goal in this set of notes. Before we get to this, we need a definition. Before that, we should have an example.

Example 1.3.1 Consider the set of vectors $\{x_1, x_2, x_3\}$ in $\mathbb{R}^3$ where $x_1 = (2,-1,5)^T$, $x_2 = (1,3,1)^T$, and $x_3 = (-1,11,-7)^T$. Let $S = \mathrm{Span}(x_1, x_2, x_3)$. If we pick just two of these vectors, say x_1 and x_2, it is clear that $\mathrm{Span}(x_1, x_2) \subseteq S$. In fact, $\mathrm{Span}(x_1, x_2)$ is a *subspace* of S. Notice that x_3 is a linear combination of the other two vectors. In particular,

$$x_3 = -2x_1 + 3x_2 . \qquad (1)$$

Therefore, every vector that is a linear combination of $\{x_1, x_2, x_3\}$ can actually be written as a linear combination of just x_1 and x_2:

$$\begin{aligned} v &= a_1x_1 + a_2x_2 + a_3x_3 \\ &= a_1x_1 + a_2x_2 + a_3(-2x_1 + 3x_2) \\ &= (a_1 - 2a_3)x_1 + (a_2 + 3a_3)x_2 \end{aligned}$$

What this means is that $\mathrm{Span}(x_1, x_2) = \mathrm{Span}(x_1, x_2, x_3)$. But we could rewrite Equation (1) as

$$2x_1 - 3x_2 + x_3 = 0 \qquad (2)$$

and see that we could solve for any one of the three vectors (since all coefficients are non-zero). It follows that

$$\mathrm{Span}(x_1, x_2, x_3) = \mathrm{Span}(x_1, x_2) = \mathrm{Span}(x_2, x_3) = \mathrm{Span}(x_1, x_3) .$$

This is because of the dependency relationship represented by Equation (2).

On the other hand, no such relationship exists between x_1 and x_2. If there was one, there would exists coefficients c_1 and c_2, not both zero, such that

$$c_1x_1 + c_2x_2 = 0\,. \tag{3}$$

We could then solve for one of the vectors in terms of the other and see that the two vectors were multiples of one another (which isn't the case). Therefore, the only way Equation (3) can hold is if $c_1 = c_2 = 0$. As is usually the case in mathematics, this important characteristic has a name.

Definition 1.3.2 Let V be a vector space. The vectors $v_1, v_2, \ldots, v_n$ are said to be ***linearly independent*** if the only solution to $c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0$ is $c_1 = c_2 = \cdots = c_n = 0$. If there is a solution in which not *all* coefficients are zero, then the vectors are ***linearly dependent***.

Example 1.3.3 (a) Are the vectors $(2,1)^T$ and $(3,2)^T$ linearly independent or dependent?

Consider $c_1\begin{bmatrix}2\\1\end{bmatrix} + c_2\begin{bmatrix}3\\2\end{bmatrix} = 0$. This implies that $2c_1 + 3c_2 = 0$ and $c_1 + 2c_2 = 0$. Solving this system of equations, we see that $c_1 = c_2 = 0$ is the only solution. So they are independent.

(b) Are the vectors $(2,-1,4)^T$, $(3,0,2)^T$ and $(10,-2,12)^T$ linearly independent or dependent?

As before, if $c_1\begin{bmatrix}2\\-1\\4\end{bmatrix} + c_2\begin{bmatrix}3\\0\\2\end{bmatrix} + c_3\begin{bmatrix}10\\-2\\12\end{bmatrix} = 0$, then we have the system of equations

$$\begin{aligned}
2c_1 + 3c_1 + 10c_3 &= 0\\
-c_1 \qquad - \;\; 2c_3 &= 0\\
4c_1 + 2c_2 + 12c_3 &= 0
\end{aligned}$$

Solving this, we have solution set $\{(-2\alpha, -2\alpha, \alpha) : \alpha \in \mathbb{R}\}$. So they are dependent.

Exercise 1.3.4 Determine whether the following vectors are linearly independent.

(a) $(-1,2)^T$, $(5,-2)^T$ in $\mathbb{R}^2$

(b) $(1,2,-1)^T$, $(0,0,2)^T$, $(4,8,2)^T$ in $\mathbb{R}^3$

(c) 1, x^2, $x^2 - 2$ in P_3

(d) $x + 2$, $x + 1$, $x^2 - 1$ in P_3

Exercise 1.3.5 Prove that any finite set of vectors that contains the zero vector is linearly dependent.

Definition 1.3.6 Let V be a vector space. The vectors $v_1, v_2, \ldots, v_n$ form a ***basis*** of V if:

(a) $v_1, v_2, \ldots, v_n$ are linearly independent, and

(b) $v_1, v_2, \ldots, v_n$ span V.

As you may have noticed, determining whether or not a set of vectors is linearly independent or linearly dependent boils down to solving a system of equations. This is related to the determinant of a matrix.

Theorem 1.3.7 Let $v_1, v_2, \ldots, v_n$ be vectors in $\mathbb{R}^n$ and let A be the $n \times n$ matrix whose i^{th} column is v_i. Then $v_1, v_2, \ldots, v_n$ are linearly independent if and only if A is nonsingular.

Exercise 1.3.8 Use Theorem 1.3.7 to show that $(-5,2,1)^T$, $(1,0,2)^T$, $(4,1,1)^T$ are linearly independent in $\mathbb{R}^3$.

Theorem 1.3.9 Let V be a vector space with $\{v_1, v_2, \ldots, v_n\}$ a spanning set. Then any collection of m vectors, where $m > n$, is linearly dependent.

Corollary 1.3.10 Let V be a vector space and suppose that $\{v_1, v_2, \ldots, v_n\}$ and $\{u_1, u_2, \ldots, u_m\}$ are both bases for V. Then $n = m$.

This last corollary tells us that the number of vectors in a basis is a fixed number for each vector space. Let's name it.

Definition 1.3.11 Let V be a vector space. The number of vectors in a basis is called the ***dimension*** of V (this can be finite or infinite). We say the trivial vector space $\{0\}$ has dimension 0.

Example 1.3.12 $\mathbb{R}^3$ is a finite-dimensional vector space of dimension 3. The vector space consisting of all polynomials, usually denoted by P, is an infinite dimensional vector space.

Theorem 1.3.13 Let V be a vector space of dimension $n > 0$. Then:

(a) Any set of n linearly independent vectors spans V.

(b) Any set of n vectors that span V are linearly independent.

Proof: (a) Let $v_1, v_2, \ldots, v_n$ be linearly independent and let v be any vector of V. The vectors $v_1, v_2, \ldots, v_n, v$ are linearly dependent. (WHY?) So there exist scalars $c_1, c_2, \ldots, c_n, c_{n+1}$, not all zero, for which $c_1 v_1 + c_2 v_2 + \ldots + c_n v_n + c_{n+1} v = 0$. Clearly c_{n+1} must be nonzero (WHY?), therefore

$$v = (-\tfrac{c_1}{c_{n+1}})v_1 + (-\tfrac{c_2}{c_{n+1}})v_2 + \ldots + (-\tfrac{c_n}{c_{n+1}})v_n .$$

Hence, $v \in \mathrm{Span}(v_1, v_2, \ldots, v_n)$. Since v was arbitrary, we see that $v_1, v_2, \ldots, v_n$ span V.

(b) Suppose $v_1, v_2, \ldots, v_n$ span V and suppose that they are linearly dependent. Then one of the vectors, say v_n, can be written as a linear combination of the others. This means that $v_1, v_2, \ldots, v_{n-1}$ also span V (WHY?). If these vectors are also linearly dependent, we can eliminate another vector. Continuing this process, we will eventually arrive at a set of k vectors ($k < n$) that (1) still span V, and (2) are finally linearly independent. But this is impossible. (WHY?) So the vectors $v_1, v_2, \ldots, v_n$ must already be linearly independent.

Exercise 1.3.14 Answer the four "WHY?"'s in the proof of Theorem 1.3.12.

Let's notice the impact of this theorem. The definition of "basis" says that two conditions must be met for a set of vectors to be a basis: they must be linearly independent and they must span the vector space. However, this theorem tells us that if we have the right number of vectors, either condition is enough. In fact, in the proof, we saw even more than that.

Theorem 1.3.15 Let V be an n-dimensional vector space ($n > 0$). Then
(a) No set of less than n vectors can span V,
(b) Any set of less than n linearly independent vectors can be extended to form a basis.
(c) Any spanning set of more than n vectors can be pared down to form a basis.

Clearly there are many bases for most vector spaces. However, many common vector spaces have a "standard basis." The standard basis for $\mathbb{R}^3$ is $\{(1,0,0)^T, (0,1,0)^T, (0,0,1)^T\}$.

Exercise 1.3.16 What do you think the standard basis for P_3 is?