CHAPTER II: Linear Transformations

<u>Section 1</u>: Definition and Examples

Definition 2.1.1 Let V and W be vector spaces. A mapping $L: V \to W$ is a ***linear transformation*** if

(a) $L(v_1 + v_2) = L(v_1) + L(v_2)$ for all $v_1, v_2 \in V$, and

(b) $L(\alpha v) = \alpha L(v)$ for all $v \in V$ and scalar α.

The two parts of this definition are sometimes combined into one condition:

$$L(\alpha v_1 + \beta v_2) = \alpha L(v_1) + \beta L(v_2).$$

Also, a word should be said about the operations. There is no guarantee that the two different vector spaces (V and W) have the same vector addition or scalar multiplication. So for example, in part (a) above, be careful to remember that the addition on the left side of the equation takes place in V and the addition on the right takes place in W.

$$L(v_1 + v_2) = L(v_1) + L(v_2)$$

Addition is in V Addition is in W

Same thing in part (b) with the scalar multiplication.

Example 2.1.2 Define $L: \mathbb{R}^2 \to \mathbb{R}^2$ by $L(x) = 3x$. Verify that this is a linear transformation.

$$L(x + y) = 3(x + y) = 3x + 3y = L(x) + L(y), \text{ and}$$

$$L(\alpha x) = 3(\alpha x) = \alpha(3x) = \alpha L(x)$$

Clearly, this can be generalized to the fact that $L(x) = kx$ for any k is a linear transformation. If k is positive, we can think of this mapping as a stretching or shrinking by a factor of k.

Example 2.1.3 Define $L: \mathbb{R}^2 \to \mathbb{R}^2$ by $L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$. Verify that this is a linear transformation.

$$L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) = L\left(\begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + y_1 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} + \begin{bmatrix} y_1 \\ 0 \end{bmatrix} = L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) + L\left(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right), \text{ and}$$

$$L\left(\alpha\begin{bmatrix}x_1\\x_2\end{bmatrix}\right)=L\left(\begin{bmatrix}\alpha x_1\\\alpha x_2\end{bmatrix}\right)=\begin{bmatrix}\alpha x_1\\0\end{bmatrix}=\alpha\begin{bmatrix}x_1\\0\end{bmatrix}=\alpha L\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right)$$

This linear transformation can be thought of as a projection onto the x_1-axis.

Example 2.1.4 Define $L:\mathbb{R}^2\to\mathbb{R}^2$ by $L\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right)=\begin{bmatrix}x_1\\-x_2\end{bmatrix}$. Verify that this is a linear transformation.

$$L\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}+\begin{bmatrix}y_1\\y_2\end{bmatrix}\right)=L\left(\begin{bmatrix}x_1+y_1\\x_2+y_2\end{bmatrix}\right)=\begin{bmatrix}x_1+y_1\\-x_2-y_2\end{bmatrix}=\begin{bmatrix}x_1\\-x_2\end{bmatrix}+\begin{bmatrix}y_1\\-y_2\end{bmatrix}=L\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right)+L\left(\begin{bmatrix}y_1\\y_2\end{bmatrix}\right)$$

$$L\left(\alpha\begin{bmatrix}x_1\\x_2\end{bmatrix}\right)=L\left(\begin{bmatrix}\alpha x_1\\\alpha x_2\end{bmatrix}\right)=\begin{bmatrix}\alpha x_1\\-\alpha x_2\end{bmatrix}=\alpha\begin{bmatrix}x_1\\-x_2\end{bmatrix}=\alpha L\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right)$$

This linear transformation has the effect of reflecting vectors about the x_1-axis.

Example 2.1.5 Define $L:\mathbb{R}^3\to\mathbb{R}^2$ by $L\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right)=\begin{bmatrix}x_1+x_3\\x_3-x_2\end{bmatrix}$. Verify that this is a linear transformation.

$$L\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}+\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}\right)=L\left(\begin{bmatrix}x_1+y_1\\x_2+y_2\\x_3+y_3\end{bmatrix}\right)=\begin{bmatrix}(x_1+y_1)+(x_3+y_3)\\(x_3+y_3)-(x_2+y_2)\end{bmatrix}=\begin{bmatrix}(x_1+x_3)+(y_1+y_3)\\(x_3-x_2)+(y_3-y_2)\end{bmatrix}$$

$$=\begin{bmatrix}x_1+x_3\\x_3-x_2\end{bmatrix}+\begin{bmatrix}y_1+y_3\\y_3-y_2\end{bmatrix}=L\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right)+L\left(\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}\right)$$

$$L\left(\alpha\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right)=L\left(\begin{bmatrix}\alpha x_1\\\alpha x_2\\\alpha x_3\end{bmatrix}\right)=\begin{bmatrix}\alpha x_1+\alpha x_3\\\alpha x_3-\alpha x_2\end{bmatrix}=\begin{bmatrix}\alpha(x_1+x_3)\\\alpha(x_3-x_2)\end{bmatrix}=\alpha\begin{bmatrix}x_1+x_3\\x_3-x_2\end{bmatrix}=\alpha L\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right)$$

Exercise 2.1.6 Define $L:\mathbb{R}^2\to\mathbb{R}$ by $L\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right)=x_2+2x_1$. Verify that this is a linear transformation.

Example 2.1.7 Let A be an $m \times n$ matrix. Define $L_A : \mathbb{R}^n \to \mathbb{R}^m$ by $L_A(x) = Ax$. Then L_A is a linear transformation since

$$L(x+y) = A(x+y) = Ax + Ay = L(x) + L(y), \text{ and}$$

$$L(\alpha x) = A(\alpha x) = \alpha(Ax) = \alpha L(x)$$

This last example is actually pretty important. On the surface, all it says is that every $m \times n$ matrix defines a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$. Why is this so important? There are certainly other linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$, right? Wrong.

Theorem 2.1.8 Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, then there exists an $m \times n$ matrix A such that $L = L_A$.

Proof: Let $\{e_1, e_2, \ldots, e_n\}$ be the standard basis for $\mathbb{R}^n$. So $e_1 = (1,0,0,\ldots,0)^T$, $e_2 = (0,1,0,\ldots,0)^T$, etc. Define A as follows: the i^{th} column of A is given by $L(e_i)$. In notation, we write this as $A = \left(L(e_1), L(e_2), \ldots, L(e_n)\right)$. Since each $e_i \in \mathbb{R}^n$ and each $L(e_i) \in \mathbb{R}^m$, this is an $m \times n$ matrix. We just need to see that $L(x) = L_A(x)$ for all $x \in \mathbb{R}^n$. Let $x = (x_1, x_2, \ldots, x_n)^T$ be an element of $\mathbb{R}^n$. We can write this as $x = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$. Applying L to this vector yields

$$\begin{aligned} L(x) &= L(x_1 e_1 + x_2 e_2 + \cdots + x_n e_n) \\ &= x_1 L(e_1) + x_2 L(e_2) + \cdots + x_n L(e_n) \\ &= \left(L(e_1), L(e_2), \ldots, L(e_n)\right) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \\ &= Ax \\ &= L_A(x) \end{aligned}$$

Ok, so is every mapping a linear transformation? Of course not.

Example 2.1.9 Define $L : \mathbb{R}^2 \to \mathbb{R}^2$ by $L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ x_2 \end{bmatrix}$. Is this is a linear transformation? No.

$$L\left(\alpha \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = L\left(\begin{bmatrix} \alpha x_1 \\ \alpha x_2 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ \alpha x_2 \end{bmatrix} \neq \alpha \begin{bmatrix} 5 \\ x_2 \end{bmatrix}$$

Example 2.1.10 Define $L:\mathbb{R}\to\mathbb{R}$ by $L(x)=x^2$. Show this is not a linear transformation.

$$L\left(x+y\right)=(x+y)^2=x^2+2xy+y^2\neq L(x)+L(y)$$

Example 2.1.11 Define $L:P_5\to\mathbb{R}$ by $L(f)=\deg(f)$. Show this is not a linear transformation.

Since $\deg(f+g)\leq\max\{\deg(f),\deg(g)\}$, $\deg(f+g)\neq\deg(f)+\deg(g)$.

Exercise 2.1.12 Let a be a fixed vector in $\mathbb{R}^2$ The mapping $L:\mathbb{R}^2\to\mathbb{R}^2$ defined by $L(x)=x+a$ is called a *translation*. Under what conditions is this a linear transformation?

Exercise 2.1.13 Determine whether the following mappings are linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^3$.

(a) $L(x)=(x_1,x_2,1)^T$
(b) $L(x)=(x_1,x_2,x_1+x_2)^T$
(c) $L(x)=(x_1,x_2,0)^T$
(d) $L(x)=(x_1,x_2,x_1{}^2+x_2{}^2)^T$

Theorem 2.1.14 Let V and W be vector spaces. We will denote the zero vectors by 0_V and 0_W respectively. If $L:V\to W$ be a linear transformation, then $L(0_V)=0_W$.

Proof: This follows from condition (b) in the definition of linear transformation if we let $\alpha=0$.

Definition 2.1.15 Let $L:V\to W$ be a linear transformation. The ***kernel*** of L is the set $\ker(L)=\{v\in V:L(v)=0_W\}$. By the previous theorem, this set is always nonempty.

Theorem 2.1.16 Let $L:V\to W$ be a linear transformation. Then the $\ker(L)$ is a subspace of V.

Proof: Let $x,y\in\ker(L)$. Then $L(x+y)=L(x)+L(y)=0+0=0$, so $(x+y)\in\ker(L)$ also. Similarly, $L(\alpha x)=\alpha L(x)=\alpha 0=0$, so $\alpha x\in\ker(L)$.

Example 2.1.17 Find the kernel of each linear transformation.
(a) L defined in Example 2.1.3.
(b) L defined in Example 2.1.4.
(c) $D:P_4\to P_4$ defined by $D(p)=p'$ (D is usually called the ***differentiation operator*** for obvious reasons).

(a) $L:\mathbb{R}^2\to\mathbb{R}^2$ is defined by $L\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right)=\begin{bmatrix}x_1\\0\end{bmatrix}$. For this to be $\begin{bmatrix}0\\0\end{bmatrix}$, x_1 must be zero. So the $\ker(L) = \{(0, x_2)^T : x_2 \in \mathbb{R}\}$.

(b) $L : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$. For this to be $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $x_1 = x_2 = 0$. So the $\ker(L) = \{(0,0)^T\}$.

(c) The zero element in P_4 is 0. The only polynomials with derivative zero are the constant polynomials (i.e. polynomials with degree 0). So $\ker(D) = P_1$.

Exercise 2.1.18 Determine the kernel of each linear transformation $L : P_3 \to P_3$.

(a) $L(p) = x \cdot p'$

(b) $L(p) = p - p'$

(c) $L(p) = p(0) \cdot x + p(1)$

Definition 2.1.19 Let $L : V \to W$ be a linear transformation and let S be a *subset* of V. The ***image*** of S is the set $L(S) = \{w \in W : w = L(v) \text{ for some } v \in S\}$. The ***range*** is the set $L(V)$.

Theorem 2.1.20 Let $L : V \to W$ be a linear transformation. If S is a subspace of V, then $L(S)$ is a subspace of W.

Proof: Since $0_V \in S$ and $L(0_V) = 0_W$, $0_W \in L(S)$ so $L(S)$ is nonempty. Let $x, y \in L(S)$. Then $x = L(v_1)$ and $y = L(v_2)$ for some $v_1, v_2 \in S$. So $x + y = L(v_1) + L(v_2) = L(v_1 + v_2)$. Since S is a subspace of V, $v_1 + v_2 \in S$. So $x + y \in L(S)$. Likewise, $\alpha x = \alpha L(v_1) = L(\alpha v_1)$, so $\alpha x \in L(S)$.

Exercise 2.1.21 A linear transformation is called ***one-to-one*** (or ***injective***) if $L(v_1) = L(v_2)$ implies $v_1 = v_2$. Show that L is one-to-one if and only if $\ker(L) = \{0\}$.

Exercise 2.1.22 Let $L : V \to W$ be a linear transformation. The image $L(V)$ is always a subspace of W .A linear transformation is called ***onto*** (or ***surjective***) if $L(V) = W$. Let D be the differentiation operator and let $S = \{p \in P_3 : p(0) = 0\}$. Show that:

(a) D maps P_3 *onto* P_2 but is not one-to-one.

(b) $D : S \to P_3$ is one-to-one, but is not onto.