CHAPTER II: Ancient Roots

Section 1: The Paradoxes of Zeno

Reading: http://www-history.mcs.st-and.ac.uk/Biographies/Zeno_of_Elea.html

In developing a sequence of topics for this course, I ran through several ideas. I enjoy topics in the history of mathematics because they are essentially stories and while the content is by far the most important component to any story, *how* it is told is of interest to me as well. I could present material chronologically, in the order things occurred, or I could group things by topic, perhaps cover differential calculus first and then integral calculus. My initial impulse was to gather progress and stories and developments through the years and build to the wonderful achievement of Newton and Leibniz. However, stories told like this, those that build to a wonderful climax, are usually most enjoyable if the reader does not know of the coming conclusion. In this case, most of the students would have heard of Newton. Many knew of Leibniz as well. Everyone would already know where everything was heading. So instead, I decided to begin there.

Like a murder mystery where the murderer is revealed in Chapter 1 and the remainder of the book is devoted to explaining how he accomplished his mission, I started with Newton and Leibniz and their remarkable contribution to mankind. In the remaining sections, we will go back through time and see what led up to their work. The obstacles that were encountered over the years will help us appreciate Newton and Leibniz even more. And while those two men were in fact creative geniuses and their work revolutionary, they are not without forebearers. Wallis and Barrow, Fermat and Descartes, Cavalieri, Kepler, and Viète all helped pave the way for the calculus during the century before Newton and Leibniz. But to really understand and appreciate their accomplishment, we need to go back in time another 2000 years…to ancient Greece.

Greece circa 500 BC must have been an amazing place to be. From most evidence, mathematics was developed around 2000 BC in the earliest civilizations in Egypt, Babylonia, China, and India. These civilizations all flourished in major river valleys (the Nile, Tigris/Euphrates, Yangtze, and Ganges respectively) and the periodic flooding by these rivers created very fertile soil well suited for a civilization, but also necessitated tools for recalculating property lines, irrigating crops, and developing a workable calendar for predicting when the flooding would come. Mathematics to the rescue. Obviously, the mathematics of these civilizations was very applied and highly empirical. There were no theorems and no abstraction. Arithmetic, geometry and even elementary trigonometry were created for the expressed purpose of meeting certain real-world needs. Once a method or algorithm was developed that seemed to work time after time, it would be adopted by more and more people and passed on. Life went on this way for hundreds of years. The Greeks were a different sort of people however. Their civilization thrived along the shores of the Mediterranean Sea from about 500 BC to 500 AD. Essentially a collection of city-states, their primary industries relied on trade routes with all other civilizations, and they proved quite adept at this and were a wealthy people. Additionally, they were relatively safe from attackers due to the mountainous terrain of their region. These factors, along with the inherited knowledge and progress from earlier nearby civilizations in Egypt and

Babylonia, perhaps help to explain what made Greece so different. But whatever the cause, the Greeks did not just work with algorithms and procedures that seemed true. They insisted on methods that could be proved, and this is essentially the birthplace of deductive reasoning and logic. The Greeks loved beauty and order. With their wealth and relative safety, they had the time to pursue less necessary endeavors such as music, art and philosophy. In this same way, Greeks saw the beauty in a logic demonstration of a truth as compared with the simple empirical verification of it. "It seems true in most of these cases." was not nearly as satisfying as "I've proved it is true in all cases." It was in this environment that Greeks such as Plato, Aristotle, Socrates, Archimedes, and Euclid made a name for themselves in mathematics and philosophy. Before all of them however, there was Zeno of Elea.

Not much is known about the man and his life (see the above reading), but his paradoxes confounded people of his day, and in fact mathematicians have wrestled with the issues he raised since then. Indeed, some issues remained unresolved until the 20th century. He wrote on many paradoxes, but a couple are especially relevant to our study of the calculus. They involve the notion of indivisibles.

Should we assume that a magnitude is infinitely divisible or that it is made up of a very large number of small indivisible "atomic" parts? The first assumption appears more reasonable to most of us, but the second assumption proved useful as well. Zeno's contribution to the dialogue was to argue with his paradoxes that both points of view make motion impossible. We will illustrate this with the following two paradoxes:

A. **The Dichotomy** Supposing that magnitudes are infinitely divisible, consider a line segment. If it is infinitely divisible, then motion is impossible, for in order to traverse the entire line segment, one must reach the midpoint, and to do this, one must first reach the one-quarter point, and to do this one must first reach the one-eighth point, and so on ad infinitum. It follows that the motion can never begin.

B. **The Arrow** Supposing that magnitudes are made up of atomic parts that cannot be subdivided, consider time. If time is composed of indivisible instants, then a moving arrow is always at rest, for at any instant it is in a fixed position. Since this is true at every instant, the arrow never moves.

Obviously, these two paradoxes caused a problem (hence the name paradox). We look around us and see that motion appears quite possible, so how do we explain this? Many explanations of his paradoxes have been given, but essentially the ancient Greeks lacked an understanding of infinity, infinite sums, and infinitesimals. For example, we now know that the sum of an infinite number of positive quantities could be finite, but this is counterintuitive and the Greeks were unaware of this. The end result of Zeno's paradoxes was that infinitesimals were simple excluded from Greek mathematics, and therefore mathematics of the civilized world. But that is not "beautiful" and is a very unsatisfying situation, so mathematicians did not leave it at that. Over the years, the immediate years following and the centuries that followed, the various attempts to explain away Zeno's paradoxes led to breakthrough after breakthrough in our understanding and our ability to tackle abstract problems.