Math 573 – History of Number Theory
Spring 2005 Notes

Dr. Brian K Heck
Nicholls State University

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INTRODUCTION

The father of number theory is indisputably Pierre de Fermat, a French lawyer and amateur mathematician of the 17th century. In 1637 he left an innocuous looking note in the margin of his copy of Bachet's translation of *Arithmetica* by Diophantus. It was near the section on Pythagorean triples, in other words, squares that can be separated into a sum of squares. The note reads as follows,

> *To divide a cube into two cubes, a fourth power into two fourth powers, or in general any power whatever into two powers of the same denomination above the second is impossible, and I have assuredly found a remarkable proof of this, but the margin is too narrow to contain it.*

We can view this note in the margin as the critical link between ancient number theory and modern number theory. It is the keystone in number theory.

Number theory began roughly in the 6th century BC with the Pythagoreans, who some might call a cult of numerologists. The famous Pythagorean Theorem is attributed to them because they studied it extensively, even though the Babylonians were familiar with the result centuries earlier. Previously, the mathematical knowledge of a people was limited to geometry and some basic arithmetic. These were developed and learned for the express purpose of practical matters such as measuring property, trade, and construction.

Around 250 AD, Diophantus of Alexandria wrote the highly influential work *Arithmetica*.

With the fall of Rome to the barbarians in 476 AD, Europe entered the so-called Middle Ages (or Dark Ages). During this time, there was very little in the way of academics.

In the 13th century Leonardo of Pisa (a.k.a. Fibonacci) was the first great European mathematician. His book *Liber abaci* is devoted to arithmetic and algebra. His accomplishments would inspire and sustain European mathematics for the next 150 years.

Following the fall of Constantinople to the Turks in 1453, many scholars fled the Turkish peninsula and landed in Italy. Italian mathematics blossomed quickly, and within 50 years was the epicenter for the rebirth of mathematics and the arts in Europe (the Renaissance). In 1575 the first Latin translation of Diophantus' *Arithmetica* appeared. In 1621 Claude Bachet published his Latin translation (with notes), and it was in a copy of this work that the Frenchman Pierre de Fermat made his famous marginal note, now known as Fermat's Last Theorem.

But not until the end of the 18th century would a mathematician be worthy of being called the successor of Fermat. In 1777, Carl Friedrich Gauss was born and the amount of mathematics he would develop is truly remarkable.

In the late 19th century, David Hilbert and Godfrey Hardy were the two preeminent number theorists. In the late 20th century, we are brought back to Fermat with the announcement by Andrew Wiles in 1995 that he had proved Fermat's Last Theorem.

CHAPTER 1

ORIGINS OF MATHEMATICS

1.1 Egypt and Babylon

Scientists speculate that man has existed for hundreds of thousands of years. But it was not until fairly late in our development as a species that we created civilization. Tools, the alphabet, money, and farming are all thought to be discovered/developed around the same time, roughly 4000 BC to 5000 BC. However, in 1937 a wolf's bone was found in the former Czechoslovakia that had evidence of counting from 30,000 BC. On this bone were 55 notches, arranged by fives, in two groups (one containing 35 notches, the other 20). This suggests that man has had the ability to count long before he could write, farm, or build. It has been argued in fact, that mathematics predates mankind. Many animal species appear to have primitive counting abilities.

However, mathematics as an intellectual endeavor developed roughly along with language and writing. The first mathematical activities were counting, measuring, surveying, and building. But that is not to say that early man did not have an interest in abstract mathematics. On a Babylonian tablet dated around 1600 BC is found evidence of Pythagorean triples, factoring, and even equation solving.

From the ancient Egyptians, we have the two oldest mathematical documents in existence. A pair of papyrus scrolls (one from around 2000 BC the other from 1700 BC) serve as essentially a manual on Egyptian mathematics.

1.2 Greek Number Theorists

With the rise of the Greek civilization, essentially just a collection of small "city-states" along the rocky shores of the eastern Mediterranean Sea, we see the beginnings of abstract learning. The people were relatively safe from attackers (due to the terrain), wealthy (prime location for trade with rest of the world), and had exposure to most knowledge of earlier civilizations (especially Egypt and Babylon). From these roots, a culture developed in which knowledge was pursued to better understand philosophical issues.

Pythagoras spent his early days traveling, spending time in Egypt, the Far East, and perhaps even a Babylonian prison. During this time, he became fascinated by numbers and their meanings/properties. In his later life, he returned to Greece, and founded a society/school/cult in which he instructed others in the "quadrivium"; namely, arithmetic, music, astronomy, and geometry. [To this the "trivium" of rhetoric, grammar, and logic was added in the Middle Ages forming the European base of a liberal arts education.] Of particular interest to the Pythagoreans were the mythical properties of the integers. They also initiated the study of amicable numbers, perfect numbers, deficient/abundant numbers, figurate numbers, and Pythagorean triples.

Alexander the Great established the city Alexandria (named for himself) and wanted it to be the center of intellectual activity. The first mathematician at the academy there was Euclid. He wrote perhaps the second most influential book of all time after the Bible. *The Elements* by Euclid has been translated into more languages than any book other than the Bible and it was the standard geometry text for the next 2000 years. Think about that.

Eratosthenes, or "beta", was a very diverse contributor to mankind. He was acclaimed as a geographer, mathematician, astronomy, historian, poet, and athlete.

The last great Greek mathematician before the "decline" of the culture was Diophantus. His text *Arithmetica* was a treatise on number theory and in was in a Latin translation that Fermat made his famous note.

1.3 The Middle and Far East

CHAPTER 2

THE EUROPEAN RENAISANCE

2.1 Leonardo de Pisa (Fibonacci)

Until this point, the Western world was in some ways one enormous civilization, comprising the Romans, Egyptians, Persians, and Greeks. Trade between nations was prevalent, and knowledge, goods, and services were easily disseminated throughout the "known world". However, with the fall of Rome in 476, the intellectual achievements of the West slowed to a trickle. The Chinese, Indians/Hindus, and Persians/Arabians took up the mantle of preserving and fostering academic pursuits. Of primary interest at the beginning this section are the Persians.

After the rise of Islam following Mohammed's flight to Medina in 622, the myriad Arabian and Persian tribes united under a single nation and purpose. This new culture quickly spread throughout the Middle East as the Moslems set out to conquer and convert the known world. At its apex, the Persian Empire reached as far east as the borders of India and as far west as Spain.

Meanwhile in Europe, the political, economic, and cultural power began a slow migration to the north. Empires ceased to exist in the way they had previously. In 1085, the Christians conquered Toledo (in present day Spain) from the Moors (Persians). This was followed by an influx of Christian scholars to the city to acquire Moslem learning.

Easily the preeminent mathematician of the Middle Ages, Leonardo was born in 1175 in Pisa where his father was involved with the mercantile business. It was in his famous work *Liber abaci* that Hindu-Arabic numerals received their strongest support to date. In fact, the text opens with the following passage:

These are the nice figures of the Indians

9 8 7 6 5 4 3 2 1

With these nine figures, and with the sign 0...any number can be written as will below be demonstrated.

In the fifteen chapters that followed, Fibonacci explained the reading and writing of these new numerals, methods of calculation, computation of square and cube roots, and the solution of linear and quadratic equations. Despite the advantages of the Hindu-Arabic numerals over their more common Roman competitors, their acceptance was not immediate. But in the fifteenth century, with the advent of printing, their forms became standardized and their use became commonplace.

While *Liber abaci* is essentially an independent investigation of arithmetic and algebra, the influence of al-Khwarizmi and Abu Kamil is unmistakable. The algebra is rhetorical and applications deal primarily with barter, partnerships, and geometrical mensuration.

Probably Fibonacci's most famous contribution to mathematics as a whole, aside from his advocacy of Hindu-Arabic numerals, is the sequence that bears his name. The sequence of numbers 1, 1, 2, 3, 5, 8, ... is perhaps the most important and most studied sequence of all time. It is formed by beginning with two consecutive ones (or in more generality any two integers), and then each successive term is the sum of the two previous terms. The sequence is produced from the following problem in *Liber abaci*:

How many pairs of rabbits can be produced from a single pair in a year if every month each pair begets a new pair, which from the second month on, becomes reproductive?

If this origin were the only "application" of the Fibonacci sequence, it would not have endured such popularity. Quite the contrary, there are numerous demonstrations and applications of the sequence in nature, the arts, and of course mathematics. In mathematics, the sequence appears in Pascal's triangle, the binomial formula, probability, the golden ratio, magic squares, and many other topics.

2.2 Italy and the Theory of Equations

By the early part of the 16th century, academic endeavors including mathematics were given increased attention. A prime example of this is the fascination with cubic and quartic equations by Italian number theorists. In 1515 Scipione del Ferro (1465-1526) algebraically solved the general cubic equation $x^3 + mx = n$. He did not publish this result, but revealed the secret to one of his students. When, in 1535, Nicola Fontana ("Tartaglia") claimed to have discovered an algebraic solution to the cubic $x^3 + px^2 = n$, del Ferro's student challenged Tartaglia to a mathematical dual. In preparation for the public contest, Tartaglia diligently applied himself and only a few days before the event found an algebraic solution to cubics lacking a quadratic term. He was victorious in the contest.

In 1545, Girolamo Cardano published the great Latin treatise *Ars magna.* In it, he revealed Tartaglia's secret to solving cubics, having wrested it from Tartaglia under a solemn pledge of secrecy. A student of Cardano denied this claim, insisting that this information was received from del Ferro. The dispute became quite virulent and Tartaglia was lucky to have survived it.

An interesting byproduct of this intense push for algebraic solutions to polynomial equations was the growing exposure of Europe to imaginary numbers. Since ancient times there were always "disallowed" numbers. We have seen zero, negative numbers, and irrational numbers all progress through this stigmatized state to eventual acceptance. However, square roots of negative numbers were still taboo and "nonsense". It was Cardano who took the first peek into a world where such numbers existed. In *Ars magna*, he posed the following problem and solution.

> *Someone says to you, divide 10 into two parts, one of which multiplied into the other shall produce 40. This is impossible to solve. Nevertheless, we shall solve it in this fashion...*

He then suggested $5+\sqrt{-15}$ and $5-\sqrt{-15}$ as the two required numbers. Indeed,

$$(5+\sqrt{-15})+(5-\sqrt{-15})=10, \text{ and}$$

$$(5+\sqrt{-15})\cdot(5-\sqrt{-15})=25-5\sqrt{-15}+5\sqrt{-15}-(-15)=40.$$

Admittedly, Cardano called this solution "puzzling" and said it was "as subtle as it is useless." Fortunately, this development was far from useless.

A seemingly paradoxical situation developed soon after Cardano's controversial publication was released. A cubic equation with three real solutions was found in which Cardano's cubic formula ran across $\sqrt{-5476}$. This stopped all calculations, and rendered this equation "unsolvable".

This trouble remained unresolved until Rafael Bombelli in 1572. In his text *Algebra,* he suggested that square roots of negative numbers could be introduced, at least temporarily, in solving cubic equations. In so doing, Bombelli was able to utilize $\sqrt{-1}$ and find the solution $x = 10$ to the equation $x^3 - 78x = 220$. This approach, called a "wild thought" by Bombelli himself, amazed mathematicians of the day. However, they were not convinced in the reality that such numbers existed for quite some time. In fact, Bombelli even said that this approach seemed to work by magic. In 1637, when René Descartes published *La Geometrie*, he referred to numbers such as $\sqrt{-9}$ as "imaginary". They would remain on the fringe of acceptance, yet outside of it, until the 18^{th} century when Leonhard Euler introduced the symbol i for $\sqrt{-1}$.

2.3 France and Prime Numbers

The first great French mathematician after the Renaissance was François Viète. Born in 1540, Viète was primarily a lawyer and a member of parliament. As his wealth increased, however, he spent more and more of his leisure time studying mathematics. In particular, he was interested in cryptography, trigonometry, and algebra.

His political life was routinely besieged by turmoil. He was allied with the Protestant Reformation in France, and therefore he had many Catholic (and powerful) enemies. He was also highly critical of the Gregorian calendar prepared by Cristof Clavis, and unleashed a long series of bitter attacks against him. This did not repair his image in the eyes of the church.

Mathematically, he is most known for his algebraic symbolism. He culled from the most respected texts of the previous 1500 years, and made many improvements. He used vowels of the alphabet for unknown quantities and consonants for known ones. Instead of a different symbol for each power of a quantity or unknown, Viète made the change to a single variable properly qualified. In other words, $x, x^2, x^3, \ldots$ would have been written as $A, A\,\text{quad}, A\,\text{cub}\ldots$. By this time other standards were becoming common in algebra. In 1489 Widman introduced the + and - signs to represent positive and negative quantities (respectively), and in 1557 Robert Recorde used the two equal length parallel lines "=" to represent equality. He explained this symbol saying: "bicause noe 2 thynges can be moare equalle."

In 1579 Viète approximated π correct to 9 decimal places by using the classical polygon method (having 393,216 sides). He also discovered the equivalent of the interesting identity

$$\pi = 2 \cdot \frac{2}{\sqrt{2}} \cdot \frac{2}{\sqrt{2+\sqrt{2}}} \cdot \frac{2}{\sqrt{2+\sqrt{2+\sqrt{2}}}} \cdots.$$

Another French number theorist of the 17th century is Marin Mersenne. A Minimite friar, Mersenne corresponded with many of the greatest mathematicians of his day. In a time when there were no research journals, Mersenne served as a disseminating body for new results. Upon his death, letters were found from 78 different mathematicians including Fermat, Huygens, Pell, Hobbes, Galileo, and Torricelli. He also regularly opened the monastery for gathering of mathematicians including Desargues, Roberval, Descartes, and both Pascals (father and son).

Although much of his fame can be attributed to this centrality to mathematical research during the century, he is most remembered for his involvement in the search for perfect numbers. Since the days of the early Greeks, mankind had been fascinated with perfect numbers. Mathematicians and non-mathematicians alike were intrigued by their curious properties. Even the theologian Saint Augustine (354-430) remarked that even though God could have created the world in one day (if he so desired), he chose to take 6 days because 6 is the first perfect number.

2.4 Pierre de Fermat

When we think of the most brilliant people in mathematics in the last 400 years, names like Rene Descartes, Carl Gauss, Isaac Newton, Gottfried Leibniz, and Blaise Pascal (among others) are sure to come to mind. Descartes discovered/created analytic geometry, Gauss was a genius in number theory, Newton and Leibniz developed the calculus (which has been called the greatest achievement of mankind), and Pascal counts the founding of probability theory among his many claims to fame. However, aside from forming an impressive roster of mathematicians, another thing all these greats have in common is a debt of gratitude to Pierre Fermat.

Fermat was born in France to wealthy and hard working parents. From an early age, Fermat showed a great deal of interest classical works in many fields such as Latin, philosophy, literature, and mathematics. However, in the early 17th century, mathematics was not a profession with employment opportunities outside of private tutoring. So while working as a lawyer, Fermat spent his free time on many mathematical problems and topics that would have enormous impact.

Fermat's true love, and the area of his most enduring mathematical contributions, was number theory. He posed many problems in the field in correspondences with Pascal, Frenicle de Bessy, Christian Huygens, Marin Mersenne, and Gilles Roberval. In many cases he claimed to have solved the stated problem, but would only give further explanation after the recipient had attempted the problem. He resisted many requests to publish his proofs, ideas, and results. He seemed genuinely disinterested in fine tuning a proof to the point where is could be published. Rather, he enjoyed jotting down a few hints or notes and then announcing the conclusion.

Aside from his voluminous collection of letters, from which many of results were retrieved, Fermat left a large collection of theorems/conjectures in the margins of his books. Five years after his death, his son Samuel brought forth a new edition of Diophantus' *Arithmetica* that contained his fathers marginal notes. It was in this edition that we first learned of Fermat's most famous theorem.

2.5 Carl Friedrich Gauss

It is unfair to a few great mathematicians to proclaim Gauss as the next great number theorist. However, in the Introduction of our notes I said (in reference to Gauss),

> But not until the end of the 18th century would
> a mathematician be worthy of being called the
> successor of Fermat.

We will begin this section by briefly discussing those being dissed, and then move on the great Carl Friedrich Gauss.

From Fermat to Gauss

A quick glance through our Famous Number Theorists Chronology (between Fermat and Gauss) yields quite an impressive list of names that would be ignored if we progressed straight from Fermat to Gauss. Isaac Newton, Jacques Bernoulli, Christian Goldbach, Edward Waring, Adrien-Marie Legendre, Joseph Fourier, and Sophie Germain are just some of those ignored heretofore, but most notably absent would be the giants Joseph Lagrange and Leonhard Euler.

Leonhard Euler is by far the most prolific mathematical author of all time. He wrote around 500 papers and books throughout his lifetime, and left volumes of work upon his death. It took the St. Petersburg Academy Journal 47 years to exhaust the information therein.

Joseph Lagrange was almost as diverse a contributor to the many branches of mathematics as was Euler. Regarding number theory, he seemed to have had a special affinity. He succeeded Euler at the Berlin Academy when the former left Prussia to return to Russia.

The Great Gauss

Carl Friedrich Gauss has an impressive if not unusual claim to fame. Legend has it that he was the last mathematician to be knowledgeable of every field of mathematics. In the 100 years after he died, mathematics grew at such a rate, and it such diverse directions, that no one since Gauss can make such an unbelievable claim. His direct contributions are monumental, and he made advancements in the fields of algebra, differential geometry, differential equations, non-Euclidean geometry, complex analysis, real analysis, group theory, topology, and of course number theory. (He also contributed to physics, mechanics, astronomy, geodesy, and magnetism) His achievements in number theory are our main focus.

It is impossible to pinpoint the single item for which Gauss is most famous. In geometry, it is the construction of the 17-gon using only a ruler and compass. In algebra, it is the first completely satisfactory proof of the Fundamental Theorem of Algebra. In number theory, it is his proof of quadratic reciprocity.

Quadratic Reciprocity (*The Short Version*)

little goal: Find all numbers a that are squares modulo a given odd prime, p.

To do this, we simply have to square each number from 1 to $p-1$, and reduce our answers modulo p. If on the other hand, we wish to

BIG GOAL: Find all odd primes p for which a given number a is a square.

then it is a very different question. However, thanks to quadratic reciprocity, it is equally easy. First a few facts,

(i) -1 is a square modulo $p \Leftrightarrow p = 4k+1$ for some k.
(ii) 2^n is a square modulo $p \Leftrightarrow (a) \quad p = 8k+1$ or $p = 8k+7$ for some k, or
$(b) \quad n$ is even.
(iii) a prime $q = 4l+1$ is a square modulo $p \Leftrightarrow p$ is a square modulo q.
(iv) a prime $q = 4l+3$ is a square modulo $p \Leftrightarrow p$ is a not a square modulo q.
(v) a product p is a square modulo $p \Leftrightarrow$ both factors are squares or both aren't.

Put this all together and we can achieve our BIG GOAL.

Example: Find all odd primes p such that 35 is a square modulo p.

$35 = 5 \cdot 7$, so it will be a square modulo p depending on whether 5 and 7 are. But 5 (which is 4(1)+1) is a square modulo p if and only if p is a square modulo 5. The only such numbers are 1 and 4. On the other hand, 7 (which is 4(1)+3) is a square modulo p if and only if p is *not* a square modulo 7. The squares modulo 7 are 1, 2, and 4. So p would need to be 3, 5, or 6 modulo 7. Putting these conditions together gives us:

35 is a square modulo p if and only if p is 1 or 4 modulo 5, and
3, 5, or 6 modulo 7.

Hence, 35 is a square modulo the primes 19, 31, 41, 59, 61, etc...

Example: Is 90 a square modulo 331?

$$\left(\frac{90}{331}\right)=\left(\frac{2}{331}\right)\left(\frac{5}{331}\right)\left(\frac{3^2}{331}\right)=(-1)\left(\frac{5}{331}\right)(+1)=(-1)\left(\frac{331}{5}\right)(+1)=(-1)\left(\frac{1}{5}\right)(+1)=(-1)(+1)(+1)=-1$$

So no.

CHAPTER 3

THE MODERN AGE

3.1 Transcendental Numbers

A number is said to be *transcendental* if it is not a root of any integral polynomial. Said another way, a number is *algebraic* if it is the root of some integral equation of the form $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$ where the coefficients are all integers. A number is therefore *transcendental* if it is not algebraic.

In 1844, Joseph Liouville found the first transcendental number. He was attempting to show that e was transcendental, but instead found a whole class of transcendental numbers, now called Liouville numbers. An example of a Liouville number is:

$$0.110001000000000000000001000...$$

It has a one in the digits corresponding to $n!$ and zeros elsewhere. In 1873 e was finally shown to be transcendental (Charles Hermite). Hermite was a very eloquent speaker and a proponent of the absoluteness of mathematical truths. Two quotations of his best illustrate this.

> There exists, if I am not mistaken, an entire world which is the totality of mathematical truths, to which we have access only with our mind, just as a world of physical reality exists, the one like the other independent of ourselves, both of divine creation.

> We are servants rather than masters in mathematics.

One other quote of Hermite I found further illustrates his dexterity with words, and is quite humorous.

> Analysis takes back with one hand what it gives with the other.

I recoil in fear and loathing from that deplorable evil: continuous functions with no derivatives.

In 1882, using the same technique as Hermite, Ferdinand Lindemann proved that π was transcendental. While this was an incredibly important discovery, Kronecker was quite unimpressed and wrote to Lindemann,

Of what use is your beautiful investigation regarding π? Why study such problems when even irrational numbers do not exist?

The transcendence of π is perhaps one of Lindemann's two greatest legacies in mathematics, the other being David Hilbert. It is no exaggeration to say that David Hilbert is one of the top five most important mathematicians of the 20th century. At the 2nd International Congress of Mathematicians in Paris, David Hilbert presented a talk entitled innocently enough "Mathematical Problems". In this talk, he outlined 23 problems facing mathematics that he believed should be researched during the coming century. His stature was such that the mathematical community went to work. The problems are as follows:

1. The Continuum Hypothesis
2. Determine the consistency of the axioms of arithmetic.
3. If two tetrahedra have equal bases and altitudes, are their volumes equal?
4. Is the shortest distance between two points a straight line?
5. The requirement of differentiability in Lie's continuous group of transformations
6. To give axiomatic treatment to mechanics and probability
7. **Irrationality and transcendence of certain numbers**
8. Problems in Prime Number Theory
   Riemann's Hypothesis, distribution of primes, Goldbach's Conjecture, the number of twin primes
9. General Law of Reciprocity in Arbitrary Number Fields
10. Determination of the solvability of Diophantine equations with rational coefficients
11. Quadratic forms with algebraic coefficients
12. Extension of Kronecker's Theorem on abelian fields
13. Impossibility of the solution of the 7th degree general equation by functions of two variables
14. Finiteness of certain systems of functions
15. Rigorous foundation of Schubert's Enumerative Calculus

16. Relative positions of the branches of a plane algebraic curve
17. Positive definite forms expressible as quotients of sums of squares of forms
18. Polyhedra in 3-space
19. Are the solutions to regular variational problems always analytic?
20. The solvability of regular variational problems with suitable boundary conditions
21. Existence of linear differential equations having a given monodromic group
22. Uniformization of analytic relations by means for automorphic functions
23. The calculus of variations

Hilbert's Seventh Problem dealt specifically with whether or not α^{β} is transcendental (or even irrational) given that α is a nonzero algebraic number and β is algebraic irrational number. In 1934 two mathematicians (Gelfond and Schneider) independently (and using different methods) proved the transcendence of α^{β}. The result has become known as the Gelfond-Schneider Theorem.

Aleksandr Gelfond (1906-1968), who was not even alive when Hilbert first posed this problem, was already well known at this time (despite his youth). In 1929, at the age of 23, he conjectured that if a_m and b_m are algebraic (for $m = 1,2,\ldots,n$) and if $\{\ln a_m\}$ are linearly independent (over the rationals), then $b_1 \ln a_1 + b_2 \ln a_2 + \cdots b_n \ln a_n \neq 0$. This was known as Gelfond's Conjecture.

In 1966, Alan Baker proved Gelfond's Conjecture, and it has since been named Baker's Theorem. Baker has taught at the prestigious University of Cambridge for the past 29 years.

Currently, research into transcendental numbers is focused on such "elementary" numbers as e^e, π^e, 2^e, π^π, and 2^π.

These numbers were unusual, but perhaps the strangest fact had not yet been revealed. In 1874, Georg Cantor proved that (1) the real numbers are uncountable, and (2) the algebraic numbers are countable. Therefore, the transcendental numbers, while appearing to be quite rare, are uncountably infinite. This was one of many results of Cantor that alienated him from the mainstream mathematical community. Kronecker in particular was unimpressed with the crazy notion that the unit interval has as many points as the plane. Cantor himself was very surprised by this result. In a letter to Dedekind, he wrote,

> Can a surface (say a square that includes the boundary) be uniquely referred to a line (say a straight line segment that includes the end points) so that for every point on the surface there is a corresponding point of the line and, conversely, for every point of the line there is a corresponding point of the surface? I think that answering this question would be no easy job, despite the fact that the answer seems so clearly to be "no" that proof appears almost unnecessary.

Three years later, upon his discovery that this was possible, Cantor wrote (again to Dedekind),

> I see it, but I don't believe it!

He also had a falling out with his close friend Magnus Mittag-Leffler. It was Mittag-Leffler who was enemies with Nobel, the result of which is there is no Nobel Prize for mathematics. In 1885, Mittag-Leffler convinced Cantor to withdraw a paper he had submitted to *Acta Mathematica* because it was, in the words of Mittag-Leffler, "... about one hundred years too soon". Cantor's reply showed his dismay.

> Had Mittag-Leffler had his way, I should have to wait until the year 1984, which to me seemed too great a demand! But of course I never want to know anything again about *Acta Mathematica.*

Cantor's continued troubles were either caused by (or caused) his many bouts with depression. He was institutionalized several times, and went through times of inactivity brought on by depressive episodes. One such time, he wrote to Mittag-Leffler,

> ... I don't know when I shall return to the continuation of my scientific work. At the moment I can do absolutely nothing with it, and limit myself to the most necessary duty of my lectures; how much happier I would be to be scientifically active, if only I had the necessary mental freshness.