II. Greece

The next great civilization after the Babylonians and the Egyptians was the Greeks. The origin of this culture is shrouded in mystery, but what is known is that they were the dominant civilization from around 500 BC to 500 AD. With the Greek civilization, essentially just a collection of small “city-states” along the rocky shores of the eastern Mediterranean Sea, we see the beginnings of abstract learning. The people were relatively safe from attackers (due to the terrain), wealthy (prime location for trade with rest of the world), and had exposure to most knowledge of earlier civilizations in Egypt and Babylon. In the field of mathematics, this love of abstract knowledge led the Greeks to embrace deductive reasoning over the previously favored empiricism.

What happened between 2000-1500 BC - when the Egyptians and Babylonians empirically “verified” procedures - and 500 BC when the rising Greek civilization embraced deduction? No one knows for certain, but there are several possible explanations. Greek society had a very philosophical nature. It was a culture that embraced philosophical inquiry, and this type of abstract thinking naturally requires logical reasoning. It stands to reason then that when they began to embrace mathematics, they would approach it with the same insistence on logical deduction of facts over empirical demonstrations of them. Of course, this partially begs the question: why were the Greeks the first to have such a philosophical nature? That remains unknown.

Another plausible explanation for this new attitude is Hellenistic love of beauty. From this point of view, the order, consistency, and completeness of a logical argument would have been much more satisfying than the shaky testimony of experiments. A third contributing factor could have been the class structure of Greek society. There were clear distinctions between the upper class intellectuals and the lower working class, and perhaps explicit demonstrations were considered the realm of the unskilled laborers. The philosophers, mathematicians, and artists were part of the upper class that generally disdained manual labor.

All of these explanations indicate a relatively *evolutionary* shift towards deductive reasoning. In other words, all these factors (and perhaps others) let to a gradual evolution from demonstrations of procedures in 1500 BC towards proofs of their validity in 500 BC. However among scholars, there are also those who hold the shift was more *revolutionary*, and this radical intellectual change came about due to some monumental event. According to this theory, the event occurred around 430 BC within a cult of numerologists known as the Pythagoreans.

Section 1: Pythagoras

Most people have heard of Pythagoras and/or his group of followers the Pythagoreans. They are undoubtedly most famous for the Pythagorean Theorem, although as we have seen, this fact was already well known long before Greece. Nevertheless, the Pythagoreans are a remarkable group of people.

Not much is know about Pythagoras himself. From some ancient summaries of Greek mathematics, it appears he was born around 572 BC on the Greek isle of Samos. In his early life, he traveled to Egypt and may have even traveled much more extensively. Upon his return home, he found Samos under the rule of a tyrant and subsequently he migrated to southern Italy. It was here, in the Greek town of Crotona, that he founded the Pythagorean school. This was essentially a secret brotherhood of men devoted to learning philosophy, mathematics, and natural sciences. They had many rites and observances that have all the markings of an ancient cult. Even once their leader died – perhaps murdered – the Pythagoreans continued on for over two centuries.

The Pythagoreans believed that everything could be explained and described using numbers (which to them meant integers). Their motto was "Everything is number." They held that each number had spiritual properties and they ruled the heavens and earth. One application of this philosophy to the natural world produced the belief among the Pythagoreans that every measurable quantity was expressible as a fraction of two integers. In other words, every number was a rational number. It should be noted that this belief was based on empirical evidence; every measurable quantity thus far had been expressible as a fraction. This was a very serious tenet of their "faith" and any dissenting opinion was not likely tolerated. For example, since $\sqrt{2}$ is a measurable quantity (in fact it is the length of the diagonal of a unit square), according to the Pythagoreans it was a rational number. Somewhere around 430 BC, however, a Pythagorean named Hippasus called this "fact" into question. He showed that $\sqrt{2}$ was irrational. In fact he *proved* it. This obviously caused quite a stir, and according to legend, the Pythagoreans took Hippasus out in a boat on the sea and threw him overboard. Despite the tragic end to this event, the Pythagoreans soon saw the utility (and indeed the *necessity*) for logical demonstrations. In fact, it is the Pythagoreans to whom credit is usually given for developing the axiomatic method, which we will talk about more in Section 2.

The Pythagoreans did a lot more than worship numbers and kill heretics. They were skilled number theorists as well. Their studies led to an appreciation and study of arithmetic (the study of numbers and their properties), and along with geometry, music, and astronomy, these four fields became the foundation of academic studies for the next 1000 years. In the ancient world, number theory was actually divided into two branches: arithmetic (the study of abstract properties of numbers) and logistic (the practical art of computing with numbers). Interestingly, today the term "arithmetic" refers to the latter term logistic and "number theory" means the former. In any case, the Pythagoreans are generally considered the first number theorists.

It is to the Pythagoreans that we attribute the discovery of ***perfect***, ***deficient***, ***abundant numbers***. Recall that a is a *divisor* of b if there exists a natural number c such that $a \cdot c = b$. Clearly, every number is a divisor of itself. A *proper divisor* is any other divisor. A number is called *perfect* if it is equal to the sum of its proper divisors, *deficient* if it is greater than this sum, and *abundant* if it is less than this sum.

Example 1: Are 20, 27, 28, and 29 perfect, deficient, or abundant?

The proper divisors of 20 are: 1, 2, 4, 5, and 10. The sum of these numbers is 22, so 20 is abundant.

The proper divisors of 27 are 1, 3 and 9 so 27 is deficient (since this sum is only 13).

The proper divisors of 28 are 1, 2, 4, 7, and 14, which sums to 28, so 28 is perfect.

The only proper divisor of 29 is 1, so it is (very) deficient.

Example 2: Are the numbers 220 and 284 perfect, abundant, or deficient?

The proper divisors of 220 are: 1, 2, 4, 5, 10, 11, 20, 22, 44, 55, and 110. These add up to 284, so 220 is abundant. On the other hand, the proper divisors of 284 are: 1, 2, 4, 71, and 142. These add up to 220, so 284 is deficient.

The numbers in this example demonstrate another interesting property. While neither equals the sum of its own proper divisors, they do equal the sum of each others. Numbers with this property are called ***amicable*** (or ***friendly***). In the interest of clarity, note that "perfect", "abundant", and "deficient" all describe a single number, while "amicable" describes a pair. A single number is not amicable.

Clearly there are many deficient numbers since every prime number is deficient, but perfect numbers seem rather rare. The first few perfect numbers are 6, 28, 496, and 8128 and these were known to the Pythagoreans, who believed that these numbers held mystical properties. In fact, the Pythagoreans believed it took the moon 28 days to complete its cycle because 28 is perfect. Along these same lines, the theologian St. Augustine (354-430 AD) once said "Six is a number perfect in itself, and not because God created all things in six days; rather, the converse is true. God created all things in six days because the number is perfect." Clearly, the study of perfect numbers has proven to be quite alluring through the years. However, even though the first four were known to the Pythagoreans, it took two millennia for the world to find the 5^{th} perfect number (which is 33,550,336). Currently there are only 47 known perfect numbers and they are all even. As we will soon see in Section 2, Euclid devised a formula for producing perfect numbers, but the ones produced will all be even and eventually it was shown that all even perfect numbers come from Euclid's formula. To this day, one of the most famous unsolved problems in number theory is whether there are *any* odd perfect numbers.

Finally, we cannot discuss the Pythagoreans without mentioning the Pythagorean Theorem. In any right triangle with sides a and b and hypotenuse c, we have $a^2+b^2=c^2$.As mentioned, this result was known to the Babylonians over a thousand years earlier, but the Pythagoreans are the believed to be the first to have proved it. Natural numbers that satisfy this equation (such as 3, 4, and 5) are called ***Pythagorean triples***. If this was the end of the story regarding this amazing theorem, it might not be such an amazing theorem. But in fact, this theorem has a very long and interesting history all by itself. Due both to its simplicity and elegance, many people have reproved this theorem in a wide variety of ways (there are well over 350 different proofs). It also is related to Fermat's Last Theorem, one of the most famous results in mathematics (and one that we will study in depth in Chapter 5).

Section 2: Euclid

When mathematicians began wanting concrete demonstrations, or proofs, they wanted to be able to derive them from other known truths, which in turn should have come from still other known truths. Clearly, this cannot proceed backwards indefinitely, so the need arose for some undeniable truths to begin the process. The ***axiomatic method*** was a rigorous process whereby things could be logically demonstrated as following from a few accepted facts, called axioms or postulates. After stating the definitions of all involved concepts, the axioms would then be stated, and then conclusions could then be drawn based on these. The goal was to have as few axioms as possible. And of course, the axioms needed to be obviously true, not needing proof themselves. The Pythagoreans are often credited with devising this idea, but it was the Greek mathematician Euclid who popularized it.

It is unfortunate that there are very few documents that remain from the Greek civilization, even fewer than the much older Egyptian and Babylonian societies. This is primarily due to the overwhelming shadow cast by *The Elements* by Euclid. David Hilbert (1862-1943) once said

> "One can measure the importance of a scientific work by the number of earlier publications rendered superfluous by it."

Indeed, when *The Elements* came out, it was so clearly the pinnacle of perfection of the deductive method that virtually every previous text was promptly discarded because no previous attempt seemed necessary any longer.

Euclid of Alexandria lived around 300 BC and very little is known about his life. What is known comes mostly from a historian named Proclus (410-485) who wrote *Commentary on Euclid Book I* in the fifth century AD. Even though he was writing about Euclid almost 800 years after the fact, he had access to many historical documents that have since been lost. In this text, Proclus writes what he calls a *Eudemian Summary* (named for the author of one of the historical documents upon which he based his summary) in which he details the evolution of Greek mathematics up to the time of Euclid. He states that Euclid collected many theorems from Eudoxus, perfected many theorems of Thaeatetus, and brought to absolute certainly the things his predecessors had only loosely shown. He also states that Archimedes (287-212 BC) mentions Euclid, and it is from these clues that historians have decided upon the time of Euclid. Whenever he actually lived, Euclid is forever remembered for writing one of the most influential books, on any subject, of all time. In fact, only the Bible has been published more than Euclid's *The Elements*. So what was this most amazing book?

First of all, it is actually 13 books. Euclid's goal was to record the whole of mathematical knowledge at the time. Each book has a demonstrated theme and is arranged logically, according to the axiomatic method. The impact of this book cannot be overstated, as it was the standard geometry textbook for about 2000 years. Think about that. It was not the originality of the work however that gave it its longevity, for as mentioned above it was compiled from earlier sources. Nor was it the perfection of rigor, for Euclid made numerous hidden assumptions, in clear violation of the axiomatic method. However, this was the first major attempt to have any degree of success deriving results from axioms and proving their validity.

Euclid began his work with 17 definitions, 5 "common notions," and 5 postulates. The difference between common notions and postulates is very subtle. Common notions are axioms that are true essentially in all branches of science whereas postulates seemed to be restricted only to axioms about the branch under consideration. Following these first principles, Euclid proceeded to derive 465 propositions. Books I through VI and XI through XIII were about geometry while Books VII through X dealt with number theory and irrationals. The 5 postulates that Euclid attempted to base all of geometry on are (stated here in my wording):

1. It is possible to draw a straight line between any two points.
2. A straight line is infinite in length.
3. It is possible to draw a circle given the center and the radius.
4. All right angles are equal.
5. If a straight line crossing two straight lines makes the interior angles on the same side less than two right angles, then the two straight lines will meet on the side of the angles which are less than two right angles.

Note that while the first four postulates are simple to state, the fifth is quite complicated. This fifth postulate (known as the Parallel Postulate) became quite a hornet's nest for mathematicians dating all the way back to Euclid himself. In fact, he does not explicitly use this postulate to prove any of his propositions until Proposition 29 in Book I. The historian Proclus even went so far as to say that the postulate "...ought even to be struck out of the Postulates altogether; for it is a theorem." The trouble with calling it a theorem is that it would then need to be included with the propositions. However, it has since been shown that the Parallel Postulate is independent of the other four postulates, and therefore could not be logically deduced from them. Perhaps this is why Euclid called it a postulate. As we shall see in Chapter VI, mathematicians eventually tried imagining a world in which the Parallel Postulate did not hold, and to their surprise things worked out quite well.

The last proposition in the ninth book is the formula Euclid found for even perfect numbers. It is here he proves: *if* $2^n - 1$ *is prime, then* $2^{n-1}(2^n - 1)$ *is perfect.* This formula gave mathematician a tool for finding perfect numbers for many years. In the 18th century, Euler showed in fact that *every* even perfect number must be of this form. We will revisit the topic of perfect numbers when we study number theory in more detail in Chapter V.

Section 3: Archimedes and Others

Aside from Pythagoras and Euclid, there were several other Greek mathematicians of note. We would be negligent if we did not mention Thales, Eratosthenes, Diophantus, and the great Archimedes.

One of the "seven wise men of antiquity", Thales is widely considered the father of deductive reasoning. He lived about the same time as the Pythagoreans (it is fairly likely that Pythagoras may have studied under him) and it is to him that the first proof is credited. In fact, he is the first man to whom specific results are attributed. Thales is credited with proving the following:

- A circle is bisected by its diameter.
- The base angles of an isosceles triangle are equal.
- The vertical angles formed by two intersecting lines are equal.
- Two triangles are congruent if they have two angles and one side equal.
- An angle inscribed in a semicircle is a right angle.

The importance of these results is not in their truth (indeed the Babylonians knew the last fact 1400 years earlier), but rather because Thales proved they were true using logical reasoning instead of simply relying on experiments to back them up.

Eratosthenes, or "beta" as he was often known, was a very diverse contributor to mankind. His nickname is rumored to have come from the "fact" that he was the second best in the known world at everything. It was intended to be an insult, but clearly that would be quite an achievement if true. He lived around 250 BC and was acclaimed as a geographer, mathematician, astronomy, historian, poet, and athlete. His two greatest claims to fame are his prime number sieve and his approximation of the circumference of the Earth. Implicit in his calculation was the fact that the Earth was round, a good 1,700 years before Christopher Columbus.

Diophantus was the last great mathematician before the decline of the Greek civilization. Around 250 AD, he wrote a treatise on number theory called *Arithmetica*. This work set Diophantus apart as a genius in algebraic number theory, and still today algebraic equations requiring rational solutions are called Diophantine equations. Diophantus and his contributions will also be revisited in later sections.

When ranking the greatest mathematicians of all time, of course a great deal of subjectivity will come in to play. It is notable therefore, that in any such list three names almost always rise to the top: Isaac Newton (1642-1727), Carl Friedrich Gauss (1777-1855), and Archimedes (287-212 BC). That someone who lived so long ago is still considered among the greatest mathematicians of all time illustrates not only brilliance of his mind but the grand scale of his influence. His achievements are far too numerous to try and detail here, but we will mention a couple of anecdotes that give a taste of his genius.

Legend has it that Archimedes invented several devices to aid the Greeks against the Romans. One such device was a large magnifying glass that set wooden ships ablaze from great distances. The Greeks also attacked enemy ships with large cranes that either dropped heavy weights or actually raised the ships themselves out of the water. To Archimedes is credited the line "Give me a place to stand, and I will move the earth."

After three years of attacking Syracuse (Archimedes' hometown), the Roman general Marcellus finally breached their defenses and stormed the town. Out of respect for Archimedes, Marcellus ordered that no one hurt the acclaimed mathematician. During the siege, Archimedes was preoccupied with a diagram drawn in the sand. When a Roman soldier came upon him, Archimedes ordered him to get out of his way. The soldier promptly stuck his spear through his chest.

Archimedes' accomplishments include early work on the integral calculus, as well as contributions to geometry, arithmetic, algebra, mechanics, astronomy, and hydrostatics. He is also considered the first to approximate π using polygons inscribed and circumscribed in circles.